Final Examination

Introduction to Topology

Great Bay University, July, 2023.

The exam consists of 7 pages, not including this cover page.
The first part of the exam consists of short questions. No partial credit will be given. The second part of the exam consists of longer questions. Partial credit will be given for correct reasoning. There are 100 points, total, on this exam.

For each $n \geq 1$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space (with the standard Euclidean metric).
Good luck!

Page	Points	Score
1	10	
2	10	
3	20	
4	20	
5	20	
6	20	
Total:	100	

Short Questions

No partial credit will be awarded. Clearly write down the choice(s) for each of the multiple-choice questions.

1. (5 points) An *open map* is one which sends open sets to open sets, a *closed map* takes closed sets to closed sets. Which of the following statements are true ?

   A. The exponential map $f\colon \mathbb{R} \to S^1, f(x) = e^{ix}$, is open.

   B. The canonical projection $p_X\colon X \times Y \to X,\ (x, y) \mapsto x$ is open.

   C. The map $h\colon \mathbb{C} \to \mathbb{C},\ h(z) = z^3$, is both open and closed.

   D. The map $k\colon \mathbb{R} \to \mathbb{R}$ is both open and closed, $k(x) = \begin{cases} 0 & \text{for } x < 0; \\ x, & \text{for } x \geq 0. \end{cases}$

   1. ____________

   Solution: A, B, C.

2. (5 points) Which of the following statements are true ?

   A. The space $X = \mathbb{R}^n \setminus \{p, q\}$ has trivial fundamental group for each $n \geq 2$; i.e., $\pi_1(X) = 0$.

   B. The torus T^2 with an open disk removed is compact.

   C. The path components of a topological space are closed subsets.

   D. $[0, 1] \times [0, 1)$ is homeomorphic to $[0, 1) \times [0, 1)$.

   E. If X or Y is contractible, then every map $f\colon X \to Y$ is homotopic to the constant map.

   2. ____________

   Solution: B, D, E.

Problems

For the following problems, show all your work. Partial credit will be awarded for correct reasoning.

3. (10 points) Suppose $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \subseteq \text{int}(X_{n+1})$ for each n. If $f\colon X \to Y$ is a function such that, for each n, the restriction $f|X_n\colon X_n \to Y$ is continuous with respect to the subspace topology on X_n, show that f itself is continuous.

Solution: Proof. For any $x \in X = \bigcup_{n=1}^{\infty} X_n$, $x \in X_n$ for some n; since $X_n \subseteq \text{int}(X_{n+1})$, $x \in \text{int}(X_{n+1})$ and hence

$$X = \bigcup_{n=1}^{\infty} \text{int}(X_n).$$

Let V be an open subset of Y, since $f|X_n$ is continuous with respect to the subspace topology on X_n,

$$(f|X_n)^{-1}(V) = f^{-1}(V) \cap X_n$$

is open for each n; that is, $f^{-1}(V) \cap X_n = U_n \cap X_n$ for some open subset U_n of X. It follows that

$$f^{-1}(V) \cap \text{int}(X_n) = U_n \cap \text{int}(X_n).$$

Thus

$$\begin{aligned} f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap \bigcup_{n=1}^{\infty} \text{int}(X_n) \\ = \bigcup_{n=1}^{\infty} (f^{-1}(V) \cap \text{int}(X_n)) \\ = \bigcup_{n=1}^{\infty} (U_n \cap \text{int}(X_n)) \end{aligned}$$

is open in X.

4. Denote by $\mathcal{S}$ the set of the capital letters in the English alphabet in a sans serif font; that is,

$$\mathcal{S} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{J}, \mathsf{K}, \mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{O}, \mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V}, \mathsf{W}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}\}.$$

(a) (5 points) Find 5 homeomorphism equivalence classes of elements of $\mathcal{S}$.

(b) (5 points) Find all the homotopy equivalence classes of elements of $\mathcal{S}$.

Solution: The homeomorphism classes are

$$\{\mathsf{A}, \mathsf{R}\}, \quad \{\mathsf{B}\}, \quad \{\mathsf{C}, \mathsf{G}, \mathsf{I}, \mathsf{J}, \mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{S}, \mathsf{U}, \mathsf{V}, \mathsf{W}, \mathsf{Z}\},$$
$$\{\mathsf{D}, \mathsf{O}\}, \quad \{\mathsf{E}, \mathsf{F}, \mathsf{T}, \mathsf{Y}\}, \quad \{\mathsf{H}, \mathsf{K}\}, \quad \{\mathsf{P}\}, \quad \{\mathsf{Q}\}, \quad \{\mathsf{X}\}.$$

The homotopy classes are

$$\{\mathsf{A}, \mathsf{R}, \mathsf{D}, \mathsf{O}, \mathsf{P}, \mathsf{Q}\}, \quad \{\mathsf{B}\}, \quad \{\mathsf{C}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{I}, \mathsf{J}, \mathsf{K}, \mathsf{L}, \mathsf{M}, \mathsf{N}, \mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V}, \mathsf{W}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}\}.$$

5. (10 points) A space is *simply-connected* if it is path-connected and has trivial fundamental group. Show that if a space X can be covered by two simply-connected open sets U, V such that $U \cap V$ is path-connected, then X is simply-connected.

Solution: Sketch of Proof. $X = U \cup V$, U, V are path-connected and $U \cap V$ is path-connected imply that X is path-connected (need a detailed proof).

Applying the van Kampen theorem, we have

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V)/N$$

for some normal subgroup N. Since $\pi_1(U)$ and $\pi_1(V)$ are trivial, we have $\pi_1(X) = 0$.

6. (10 points) The following is a basic fact:

> Let A be a contractible subcomplex of a CW-complex X, then the quotient map $X \to X/A$ is a homotopy equivalence.

Let Γ be a connected finite graph, which is a CW-complex of dimension 1. Show that the rank of $H_1(\Gamma)$ is $1 - \chi(\Gamma)$, where $\chi(\Gamma)$ is the Euler characteristic of Γ.

Solution: Sketch of Proof. By the quoted statement in the problem, there is a homotopy equivalence

$$\Gamma \simeq \bigvee_m S^1,$$

where m is the number of loops in Γ. Thus

$$H_1(\Gamma) \cong \pi_1(\bigvee_m S^1) \cong \mathbb{Z}^m, \quad \operatorname{rank}(H_1(\Gamma)) = m.$$

By the Euler characteristic formula,

$$\chi(\Gamma) = 1 - \operatorname{rank}(H_1(\Gamma)) = 1 - m.$$

7. (10 points) Let $f\colon S^n \to S^n$ be a map satisfying $f(x) = f(-x)$ for all x (f is called an *even map*). Show that the degree of f is zero if n is even.

Solution: $f(x) = f(-x)$ implies that f factors as the composition

$$f\colon S^n \xrightarrow{p} \mathbb{R}P^n \xrightarrow{\tilde{f}} S^n.$$

Thus $f_* = \tilde{f}_* \circ p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n) \to H_n(S^n)$. Since $H_n(\mathbb{R}P^n) = 0$ if n is even, we get $f_* = 0$ and thus f has degree 0.

8. Compute the singular homology groups $H_i(X)$ of the following spaces X:

   (a) (5 points) X is the space obtained from the torus T^2 with 3 open disks removed.

   (b) (5 points) $X = K\sharp T^2$ is the connected sum of the Klein bottle K and the torus T^2.

   (c) (5 points) $X = \mathbb{R}P^n/\mathbb{R}P^2$ for $n > 2$, where $\mathbb{R}P^n$ is the real projective space of dimension n.

   (d) (5 points) $X = S^2/\{N, S\}$ is the quotient space of S^2 by identifying its north and south poles N, S.

Solution: Sketch of Solutions.

(a) $X \simeq S^1 \vee S^1 \vee S^1 \vee S^1$.

(b) $K\sharp T^2 \cong P^2\sharp P^2\sharp T^2 = P^2\sharp(3P^2) = 4P^2$.

(c) $X = \mathbb{R}P^n/\mathbb{R}P^2 = e^0 \cup e^3 \cup \cdots \cup e^n$. For each $m < k \leq n$, the k-skeleton of X is $X^k = \mathbb{R}P^k/\mathbb{R}P^m$. Consider the composition

$$S^{k-1} \xrightarrow{\varphi_k} X^{k-1} \xrightarrow{q} X^{k-1}/X^{k-2} = \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1},$$

where $varphi_k$ is the attaching map of a k-cell of X. By similar arguments in the computation of $H_i(\mathbb{R}P^n)$, the degree of the composition $q\varphi_k$ has degree $1 + (-1)^k$. Thus we get the cellular chain complex of X:

$$0 \to C_n = \mathbb{Z} \xrightarrow{1+(-1)^n} C_{n-1} = \mathbb{Z} \to \cdots \xrightarrow{2} C_3 = \mathbb{Z} \to 0.$$

Therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z}, & k = 0 \text{ or } k = n \text{ is odd;} \\ \mathbb{Z}/2, & 3 \leq k < n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let A be a great arc in S^2 connecting N and S. Then A is contractible and hence there is a homotopy equivalence

$$X \simeq X/A \simeq S^2 \vee S^1.$$

9. (10 points) Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be an exact sequence of abelian groups. Show the following statements are equivalent:

(1) There is a homomorphism $p\colon B \to A$ such that $p \circ i = id\colon A \to A$.

(2) There is a homomorphism $s\colon C \to B$ such that $j \circ s = id\colon C \to C$.

(3) There is an isomorphism $\varphi\colon B \to A \oplus C$ making the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "j"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i_1"] & A \oplus C \arrow[r, "p_2"] & C \arrow[r] & 0 \end{tikzcd},$$

where $i_1(a) = (a, 0)$ and $p_2(a, c) = c$.

Solution: Clear.

10. Let X be a CW-complex.

(a) (5 points) Show $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{s_0\})$ for all i.

(b) (5 points) Show $H_i(X \times S^n, X \times \{s_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{s_0\})$ by the relative Mayer-Vietoris sequence.

It follows that there is an isomorphism

$$H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$$

for all i and n, where $H_i = 0$ for $i < n$ by definition.

Solution: (a) Consider the homology long exact sequence for the pair $(X \times S^n, X \times \{s_0\})$. Note that the composition

$$X \times \{s_0\} \xrightarrow{i_0} X \times S^n \xrightarrow{r} X \times \{s_0\}$$

is the identity. By the problem 9 we then get a *split* short exact sequence for each i:

$$0 \to H_i(X \times \{s_0\}) \xrightarrow{(i_0)_*} H_i(X \times S^n) \xrightarrow{j_*} H_i(X \times S^n, X \times \{s_0\}) \to 0, \ \ r_*(i_0)_* = id.$$

(b) Let

$$A = X \times D^n_+ \simeq X \times \{s_0\}, \quad B = X \times D^n_- \simeq X \times \{s_0\},$$

then $X = A \cup B$ and $A \cap B = X \times S^{n-1}$. Applying the Mayer-Vietoris exact sequence for *relative homology groups*, we then get an isomorphism

$$\partial\colon H_i(X \times S^n, X \times \{s_0\}) \xrightarrow{\cong} H_{i-1}(X \times S^{n-1}, X \times \{s_0\}), \ \forall\, i.$$

Thus $H_i(X \times S^n, X \times \{s_0\}) \cong H_{i-n}(X \times S^0, X \times \{s_0\}) \cong H_{i-n}(X)$, where the last isomorphism is obtained by excising $X \times \{s_0\}$.

End of exam.