Lecture 0. Euler's formula for polyhedra.

— What's Topology?

— Topology is a mathematical subject that studies properties of <u>geometric objects</u> that are <u>preserved</u> under <u>continuously transformations</u>.

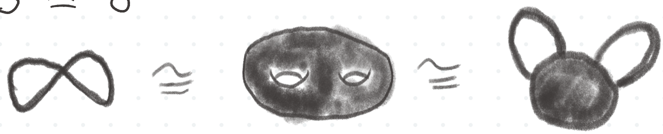
Topology is a qualitative geometry.

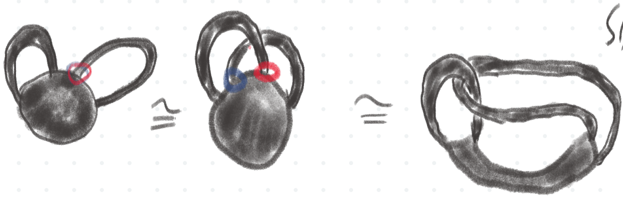
If two geometric objects A and B can be continuously transform from one to the other, then we say A and B are topologically equivalent. Denote by $A \cong B$.

Examples. ① $A \cong R$, $D \cong O$, $E \cong F \cong T \cong Y$

$B \cong 8$

②





③ A knot is an <u>embedding</u> of S^1 into $\mathbb{R}^3$

isotopy

Euler's formula for polyhedra (1750')

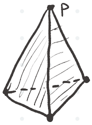
<u>polyhedron</u>: A polyhedron P is a collection of finitely many plane polygons that fit together nicely:

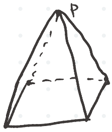
(i) If two plane polygons meet, then they meet at one edge.

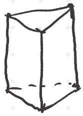
(ii) For each edge, there are exactly two polygons containing it

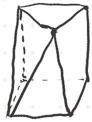
(iii) Given a vertex p of P, the polygons containing p form a piece of surface of P around p.

Examples

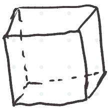








$v-e+f$ $\quad 4-6+4=2 \quad 5-8+5=2 \quad 6-9+5=2 \quad 7-10+$



$8-12+6=2$

Given a polyhedron P, $v=\#\{\text{vertices of } P\}=V(P)$

$$e=\#\{\text{edges of } P\}=e(P)$$

$$f=\#\{\text{faces of } P\}=f(P)$$

$\chi(P)=v-e+f$ is called the <u>Euler characteristic</u> of P.

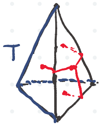
Theorem (Euler) Let P be a polyhedron such that

(a) Any two vertices of P can be connected by a chain of edges.

(b) Any loop in P that is made of line segments in P seperates P into two pieces.

Then $\chi(P)=2$.

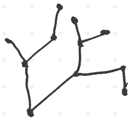
proof. By graph theory.



(connected) graph: consits of vertices and edges, and any two vertis can be connected by a chain of edges.

A tree T is a graph that doesn't contain loops.

Fact: $v(T) - e(T) = 1$



Fact: every connected graph contains a maximal tree.

The assumption (a) tells that the vertices and edges of P form a graph.

Let T be a tree on P that contains all vertices of P, and partial edges. Construct its dual graph $\mathcal{T}$:

(i) For every face f of P corresponds to a vertex of $\mathcal{T}$.

(ii) Two vertices of $\mathcal{T}$ are connected by an edge if the original faces meet in a common edge that is not in T

By construction,

$$v(\mathcal{T}) = f(P)$$

$$e(\mathcal{T}) + e(T) = e(P)$$

$$v(T) = v(P)$$

$$T \cap \mathcal{T} = \phi.$$

$$\chi(P) = v(P) - e(P) + f(P) = v(T) - e(T) - e(\mathcal{T}) + v(\mathcal{T})$$
$$= 1 + v(\mathcal{T}) - e(\mathcal{T})$$

$\therefore$ The proof completes by showing that $\mathcal{T}$ is a tree.

Assume that there is a loop in $\mathcal{T}$. By (b), the loop seperates P into two pieces, and these two pieces contain at least one vertex of P.

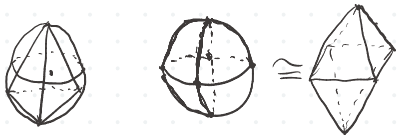
There exist a chain of edges in T that connects these two vertices.
It follows that the chain intersects the loop.

However, we have $T \cap \Gamma = \phi$. Thus there are edges of the chain lying outside of T, contradiction.

$\square$

Therefore Γ is a tree.

Theorem. If P and K are two polyhedra, then $\chi(P) = \chi(K)$.



$\underline{\chi(S^2) = 2}$

Application: If P and K are two orientable surfaces, then

$$P \cong K \iff \chi(P) = \chi(K).$$

Lecture 1. Topological Spaces $\qquad \mathbb{R}, \mathbb{R}^n$

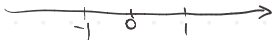
A set X together with a collection τ of subsets of X, called <u>open sets</u>, satisfying the following ~~three~~ axioms:

(1) $\phi, X \in \tau$

(2) $\forall A_\alpha \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$

(3) $A_1, A_2 \in \tau \Rightarrow A_1 \cap A_2 \in \tau$.

Example: $\mathbb{R}$ $\quad \tau = \{(a,b) \mid a<b\}$ $\quad \tau(\mathbb{R}^n) = \{ B(x,r) = \{ y \in \mathbb{R} \mid |y-x|<r \} \mid x \in \mathbb{R}^n, r>0 \}$



The pair (X, τ) is called a <u>topogical space</u>.

Let (X,τ) be a topological space. A subset $U \subseteq X$ is <u>closed</u> if $U^c = X - U \in \tau$.

Review: De Morgan's laws: $\left(\bigcup_\alpha A_\alpha\right)^c = \bigcap_\alpha A_\alpha^c$ $\quad$ 并之补=补之交

$\left(\bigcap_\alpha A_\alpha\right)^c = \bigcup_\alpha A_\alpha^c$ $\quad$ 交之补=补之并

Distributive law. $A \cup \left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha (A \cup B_\alpha)$

$A \cap \left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha (A \cap B_\alpha)$

Exercise. Define closed sets on X by 3 axioms

- Every set X has at least two topologies.

trivial topology: $\tau_0 = \{\phi, X\}$

discrete topology: $\tau_\infty = \{\text{subsets of } X\}$

$\mathcal{T}_0 \subseteq \mathcal{T}_\infty$

Every topology $\mathcal{T}$ on a set X satisfies $\mathcal{T}_0 \subseteq \mathcal{T} \subseteq \mathcal{T}_\infty$

Given a set X and two topologies $\mathcal{T}_1, \mathcal{T}_2$ on X. we say that

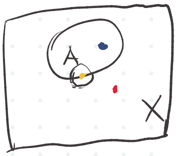
$\mathcal{T}_1$ is <u>coarser</u> than $\mathcal{T}_2$, or equivalently, $\mathcal{T}_2$ is <u>finer</u> than $\mathcal{T}_1$,

if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

A <u>neighborhood</u> U_x of $x \in X$ is an open set containg x.

($x \in O_x \subseteq U_x \subseteq X$, Somebody likes this definition)

- Let $A \subseteq X$ be a subset of a topological space X, and let $x \in X$.



There are exactly 3 possibilities for the positions of x relative to A:

(1) There exists a $U_x \subseteq A$.

(2) There exist a $U_x \subseteq A^c = X - A$

(3) Every U_x satisfies $U_x \cap A \neq \phi$, $U_x \cap A^c \neq \phi$

<u>Defs</u> <u>interior point</u>. $x \in A$ if $\exists\ U_x \subseteq A$.

<u>$x \in \partial A$</u> if every $U_x \subseteq X$ satistisfies $U_x \cap A \neq 0$, $U_x \cap A^c \neq \phi$

x is a <u>limit point</u> of A if every $U_x \cap A \neq \phi$.

$x_1, x_2, \cdots, x_n, \cdots \to x$

The closure $\bar{A}$ is the set of all limit points of A.

The interior $\mathrm{int}(A) = A^\circ$ is the set of all interior point

There hold formulas.

$$\bar{A} = \partial A \amalg \mathrm{int}(A)$$

$$X = \mathrm{int}(A) \amalg \partial A \amalg \mathrm{int}(A^c)$$

=Proposition

Prop. Let A be a subset of X. The followings hold:

(1) $\mathrm{int}(A)$ is open.

(2) $\overline{A}$ is closed.

(3) A is open iff (= if and only if) $A = \mathrm{int}(A)$.

(4) A is closed iff $A = \overline{A}$.

proof. Exercise.

Topology basis

Let (X, τ) be a topological space. A collection $\mathcal{B}$ of open sets of X ($\mathcal{B} \subseteq \tau$) is called a basis for τ if every open set of X is a union of subsets of $\mathcal{B}$.

If $\mathcal{B}$ is a basis for τ, then

(1) $X = \bigcup_\alpha B_\alpha$, $B_\alpha \in \mathcal{B}$

(2) $\forall B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 = \bigcup_\lambda B_\lambda$, $B_\lambda \in \mathcal{B}$.

prop. Let $\mathcal{B}$ be a collection of subsets of X satisfying (1), (2) above. Then $\mathcal{B}$ is a basis for a topology on X.

Here a "topology" is the topology generated by $\mathcal{B}$, denoted by $\tau_{\mathcal{B}}$.

$U \subseteq X$ is open $\Leftrightarrow$ $U = \bigcup_\alpha B_\alpha$, $B_\alpha \in \mathcal{B}$.

proof. Exercise.

$$U \cap V = \left(\bigcup_\alpha B_\alpha\right) \cap \left(\bigcup_\lambda B_\lambda\right) = \bigcup_{\alpha,\lambda} \underline{B_\alpha \cap B_\lambda} \in \tau_{\mathcal{B}}. \quad \square$$

metric spaces

eg. $\mathbb{R}^n$

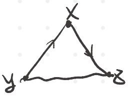
distance

Def. A set X is called a metric space if there is a metric $d: X\times X \longrightarrow \mathbb{R}$ satisfying the following 3 axioms.

(i) $d(x,y)\geq 0$, and $d(x,y)=0$ iff $x=y$.

(ii) $d(x,y)=d(y,x)$; or equivalently $dT=d$, $T: X\times X \to X\times X$, $T(x,y)=(y,x)$.

(iii) $d(x,y)+d(y,z)\geq d(x,z)$.



Example: $(\mathbb{R}^n, d)$: $d_1(x,y)=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$

$$d_2(x,y)=\sum_{i=1}^{n}|x_i-y_i|$$

$$d_3(x,y)=\max_{1\leq i\leq n}\{|x_i-y_i|\}$$

Define $B(x,r)=\{y\in X \mid d(x,y)<r\}$, $x\in X$, $r>0$.

Prop. Let (X,d) be a metric space. Then the collection $\mathcal{B}=\{B(x,r)\mid x\in X, r>0\}$ is a basis for a topology on X. the generating topology is called the metric topology and is denoted by $\mathcal{T}_d$.

proof. It suffices to show that $\forall$ x_1, x_2 two different points in X, and $r_1, r_2>0$,

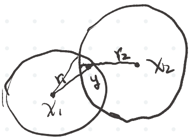
$$B(x_1,r_1)\cap B(x_2,r_2)=\bigcup_{x,r} B(x,r).$$

$\forall\, y\in B(x_1,r_1)\cap B(x_2,r_2)$,

(Claim. For $s<r-d(x,y)$, there holds $B(y,s)\subseteq B(x,r)$.)

take $s<\min\{r_1-d(y,x_1), r_2-d(y,x_2)\}$, then by the claim,

we have $B(y,s)\subseteq B(x_1,r_1)\cap B(x_2,r_2)$.



$$\bigcup_{y,s} B(y,s)=B(x_1,r_1)\cap B(x_2,r_2). \quad \square$$

Exercise. Show the above claim.

Subspace

prop. Let (X, τ) be a topological space, and let $A \subseteq X$ be a subset.

Then A inherits a topology from τ.

proof. Let $\tau_A = A \cap \tau = \{A \cap U \mid U \in \tau\}$.

Check that τ_A is a topology on A. $\square$

The topology τ_A is called the <u>subspace topology</u>. $(A, \tau_A) \subseteq (X, \tau)$.

- If $\mathcal{B}$ is a topology basis for (X, τ), then $\mathcal{B}_A = A \cap \mathcal{B}$ is a basis for τ_A.

- If (X, d) is a metric space and $A \subseteq X$, then

$\tau = \tau_d$. $\quad d_A : A \times A \xrightarrow{i \times i} X \times X \xrightarrow{d} \mathbb{R}$

is a metric on A.

Exercise. $\tau_{d_A} = \tau_A$: the induced metric topology by τ_{d_A} coincides with the subspace topology τ_A.

- Let $X \subseteq Y$ be a subspace and let $A \subseteq X$ be a subset. Then

(1) If X is open (resp. closed), then $A \subseteq X$ is open (resp. closed) implies $A \subseteq Y$ is open (resp. closed).

(2) $\bar{A}$ in X $= (\bar{A}$ in $Y) \cap X$.

proof. Exercise.

product Spaces

Let (X, τ_1) and (Y, τ_2) be topological space. The Cartesian product $X \times Y$ inherits a topology, called the product topology.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$$\tau = \bigcup_{\alpha, \beta} U_\alpha \times V_\beta, \quad U_\alpha \in \tau_1, \; V_\beta \in \tau_2.$$

$\bigcup (a,b) \times (c,d)$

If $\tau_1 = \tau_{\mathcal{B}_1}$, $\tau_2 = \tau_{\mathcal{B}_2}$, then $\tau = \tau_{\mathcal{B}_1 \times \mathcal{B}_2}$