Lecture 2. Topological properties of spaces.

1. Continuous functions = maps.

(ε, δ) arguments: $f: \mathbb{R} \to \mathbb{R}$ is continuous if $\forall \varepsilon > 0$, $\overset{=f(x_1)}{y_1}, \overset{=f(x_2)}{y_2} \in \mathbb{R}$
$\exists \delta > 0$, $x_1, x_2 \in \mathbb{R}$, such that
$|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| < \varepsilon$.

<u>Defn.</u> A function $f: X \to Y$ between two topological spaces X, Y is <u>continuous</u> if the preimage of open sets in Y is open in X.

prop. Let $f: X \to Y$ be a function. The followings are equivalent:

(i) f is continuous
(ii) The preimage of closed sets in Y is closed in X.
(iii) $\forall y \in Y$ and V_y (nbhd of y), $f^{-1}(V_y)$ is a nbhd of x, $f(x) = y$.
(iv) $f(\overline{A}) \subseteq \overline{f(A)}$ for any set $A \subseteq X$
(v) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$.

proof. (i), (ii), (iii) are equivalent: <u>exercise</u>

Exercise. Let $f: X \to Y$ be a function and $\mathcal{B}$ is a topology basis of Y. Show that f is continuous iff $\forall B \in \mathcal{B}$, $f^{-1}(B) \subseteq X$ is open.

Examples. ① $\mathbb{1}_X: X \xrightarrow{1_X / id} X$, $x \mapsto x$, is continuous.

② $e: X \to \{x_0\}$, is continuous

③ the diagonal map $\Delta: X \to X \times X$, $\Delta(x) = (x, x)$, is continuous.

④ if $A \subseteq X$ is a subspace, then the inclusion map $i: A \to X$, $i(a) = a$, is continuous.

⑤ $X \xrightarrow{i_1} X \times Y$, $Y \xrightarrow{i_2} X \times Y$ are continuous.
$x \mapsto (x, y_0)$

⑥ $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are continuous.

prop. Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be continuous functions.

Then the composition $g\circ f: X \longrightarrow Z$ is continuous.

proof. $\forall\, U \subseteq Z$ be open, $(g\circ f)^{-1}(U) = f^{-1}\circ g^{-1}(U) \subseteq X$ is open. $\square$

Applications. ① Let $A \overset{i}{\hookrightarrow} X$ be the inclusion map and $f: X \longrightarrow Y$ be a map.

Then $f|_A = f\circ i: A \xrightarrow{i} X \xrightarrow{f} Y$ is continuous.

②
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_1 = p_1 f"'] & X\times Y \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

$f(z) = (p_1 f(z), p_2 f(z))$
$= (f_1(z), f_2(z))$

$Z \xrightarrow{f=(f_1,f_2)} X\times Y$ is continuous iff f_1 and f_2 are continuous.

$\Rightarrow$ clear

$\Leftarrow$ $Z \xrightarrow{\Delta} Z\times Z \xrightarrow{f_1\times f_2} X\times Y$ is f.

$z \longmapsto (z,z) \longmapsto (f_1(z), f_2(z)) = f(z)$

$f = (f_1\times f_2)\circ \Delta$. $\square$

or $f^{-1}(U\times V) = f_1^{-1}(U)\cap f_2^{-1}(V)$

Exercise. Let $f, g: X \longrightarrow \mathbb{R}$ be continuous map. Show that

(1) $f\pm g: X \longrightarrow \mathbb{R}$, $(f\pm g)(x) = f(x)\pm g(x)$, is continuous.

(2) $f\cdot g: X \longrightarrow \mathbb{R}$, $(f\cdot g)(x) = f(x)\cdot g(x)$, is continuous.

($X \xrightarrow{\Delta} X\times X \xrightarrow{f\times g} \mathbb{R}\times\mathbb{R} \xrightarrow{\pm} \mathbb{R}$)

Theorem (gluing lemma, pasting lemma).

Let $X = \bigcup_{i=1}^n X_i$, and let $f: X \longrightarrow Y$ be a function.

If each restriction $f_i = f|_{X_i}: X_i \longrightarrow Y$ is continuous,

and each X_i is closed (resp. open), then f is continuous. $(f = \bigcup_{i=1}^n f_i)$.

proof $\forall\, V \subseteq Y$ is closed, $f^{-1}(V) = X\cap f^{-1}(V) = (\bigcup_{i=1}^n X_i)\cap f^{-1}(V)$

$= \bigcup_{i=1}^n X_i \cap f^{-1}(V)$

$= \bigcup_{i=1}^n f_i^{-1}(V) \subseteq X$ is closed.

Thus f is continuous.

- If each X_i is open, the statement is still true if $n=\infty$.

Homeomorphism. $f: X \to Y$ is a homeomorphism if

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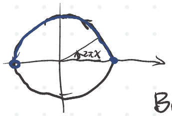
(i) f is continuous

(ii) f is a bijection: the inverse f^{-1} exists.

$f: X \xrightarrow{\cong} Y$ (iii) The inverse $f^{-1}: Y \to X$ is continuous.

Example: ① $f(x) = e^{i2\pi x}: [0,1) \longrightarrow S^1$ is continuous.



$g: S^1 \longrightarrow [0,1)$, $g(z = e^{i\theta}) = \frac{\theta}{2\pi}$. $0 \leq \theta < 2\pi$

$fg = \mathbb{1}$, $gf = \mathbb{1}$.

But g is not continuous.

$g^{-1}([0,\frac{1}{2})) =$  $\subseteq S^1$ is not open.

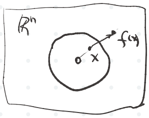
② $f: (0,1) \xrightarrow{\cong} \mathbb{R}$, $f(x) = \frac{x}{1-x}$; $g: \mathbb{R} \longrightarrow (0,1)$, $g(y) = \frac{y}{1+y}$.

f is a homeomorphism

③ $B(0,1) = \{x \in \mathbb{R}^n \mid |x| < 1\} \xrightarrow{\cong} \mathbb{R}^n$

④ $\mathbb{R}^n \setminus \{0\} \xrightarrow[\cong]{f} \mathbb{R}^n \setminus D^n$, $D^n = \overline{B(0,1)}$

$f(x) = x + \frac{x}{\|x\|}$



⑤ Stereographic projection: $S^2 \setminus \{N\} \xrightarrow[\cong]{\phi} \mathbb{R}^2$

$\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

A property P is called a topological property of a space if it is preserved by homeomorphisms of the space.

Hausdorff. A space X is Hausdorff if it satisfies the T_2 axiom:

$\forall\, x \neq y \in X$, $\exists$ nbhds U_x and U_y s.t. $U_x \cap U_y = \phi$.

Example: metric spaces are Hausdorff

prop. Let $f: X \to Y$ be an injective map. If Y is Hausdorff, then so is X.

proof. $\forall\, x_1 \neq x_2 \in X$. $f(x_1) \neq f(x_2)$. Since Y is Hausdorff,

$\exists\, V_{f(x_1)} \cap V_{f(x_2)} = \phi$.

$U_{x_1} = f^{-1}(V_{f(x_1)})$, $U_{x_2} = f^{-1}(V_{f(x_2)})$.

Then $U_{x_1} \cap U_{x_2} = \phi$. $\square$

Cor. Hausdorff property is a topological property.

prop. The followings hold:

(1) points of a Hausdorff space are closed

(2) Subspaces of a Hausdorff space are Hausdorff.

(3) product spaces of two Hausdorff spaces are Hausdorff.

proof. Exercise.

prop. Let $f: X \to Y$ be a map with Y Hausdorff. Then

(1) The graph $G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is closed.

(2) Let $g: X \to Y$ be another map. Then $eq(f,g) = \{x \in X \mid f(x) = g(x)\}$ is a closed subset of X. (exercise)

proof. (1) $Z = X \times Y - G_f$. $\forall\, (x, y) \in Z$. $y \neq f(x)$.

Since Y is Hausdorff, there exist V_y and $V_{f(x)}$, s.t. $V_y \cap V_{f(x)} = \phi$.

$U_x = f^{-1}(V_{f(x)})$ is a nbhd of x. Then $U_x \times V_y \subseteq Z = X \times Y - G_f$.

$\therefore Z$ is open $\square$

Connectedness.

A topological space X is connected if one of the following equivalent conditions holds:

(1) X cannot be expressed as a union of two disjoint non-empty open sets.

(2) X - - - - - - - - - - - - - - - - closed sets.

$\boxed{(3)}$ The subsets of X that are both open and closed are ϕ, X.

Exercise. Show the above three statements are equivalent.

prop. Let $f: X \to Y$ be a surjective map with X connected.

Then $Y = f(X)$ is connected. (the image of connected space is connected)
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proof. Let V be an open and closed subset of Y.

$f^{-1}(V)$ is both open and closed as a subset of X.

Since X is connected, $f^{-1}(V) = \phi$, or $f^{-1}(V) = X$.

Thus $V = \phi$ or $V = f(X) = Y$.

That is, Y is connected. $\square$

Cor. Connectedness is a topological property.

- Want. If X and Y are connected, then $X \times Y$ is connected.

$(x,y) \in (X \times \{y\}) \cup (\{x\} \times Y)$

$$X \times Y = \bigcup_{x \in X,\, y \in Y} (X \times \{y\} \cup \{x\} \times Y).$$

Lemma. If $A \subseteq X$ is both open and closed, and $B \subseteq X$ is connected, then either $A \cap B = \phi$, or $B \subseteq A$.

proof. Assume that $A \cap B \neq \phi$.

$\phi \neq A \cap B \subseteq B$ is both open and closed.

Since B is connected, $A \cap B = B$, $B \subseteq A$. $\square$

Theorem. Let $\{X_\alpha\}$ be a collection of connected subspaces of X.

If $X_\alpha \cap X_\beta \neq \phi, \forall \alpha, \beta$, then $Y = \bigcup_{\alpha,\beta} X_\alpha \subseteq X$ is connected.

proof. Let $V \subseteq Y$ is both open and closed. Then $V \cap X_\alpha \subseteq X_\alpha$ is both open and closed. Since X_α is connected, either $V \cap X_\alpha = \phi$ or $V \cap X_\alpha = X_\alpha$.

Suppose that $V \neq \phi$, and $x \in V$. $\exists \beta_0$ s.t. $x \in V \cap X_{\beta_0}$, $X_{\beta_0} \subseteq V$.

Since $X_\alpha \cap X_{\beta_0} \neq \phi$, we have $V \cap X_\alpha \neq \phi$, $X_\alpha \subseteq V$ for $\forall \alpha$.

Thus $Y = \bigcup_\alpha X_\alpha \subseteq V$, and therefore $Y = V$, is connected. $\square$

Cor. If X and Y are connected spaces, then $X \times Y$ is connected. $\square$

Cor. $\forall x \in X$, $C(x) = \bigcup_\alpha X_\alpha$, X_α are connected spaces that contain x.

Then $C(x)$ is connected, and $\forall y \in C(x)$, $U_y \subseteq C(x)$.

$C(x)$ is the largest connected subset that contains points of $C(x)$,

We call $C(x)$ the connected component of X containg x.

Exercise. Connected components of X are closed subsets. ($\bar{C}$ is connected if C is connected).

Fact. $A \subseteq \mathbb{R}$ is connected iff A is an interval.

Path-connectedness.

A space X is path connected if every two points x, y ~~of~~ X can be joined by a path: $\exists\ \gamma: [0,1] \longrightarrow X$, $\gamma(0) = x$, $\gamma(1) = y$.

$[0,1] \overset{\cong}{\to} [a,b] \xrightarrow{\gamma} X$

$a < b$

Prop. Let $f: X \to Y$ be a surjective map with X path-connected, then Y is path-connected.

proof. $\forall\, y_1 \neq y_2 \in Y$, let $y_1 = f(x_1)$, $y_2 = f(x_2)$, then $x_1 \neq x_2$.

Since X is path-connected, $\exists\, \gamma: [0,1] \to X$, s.t. $\gamma(0) = x_1$, $\gamma(1) = x_2$.

Then $f \circ \gamma: [0,1] \to X \to Y$ is a path connecting y_1 and y_2.

Thus Y is path-connected. $\square$

Cor. Path-connectedness is a topological property.

prop 1. Path-connected spaces are connected.

proof. Let X ~~be~~ be path-connected and let $A \subseteq X$ is both open and closed.

Assume $A \neq \emptyset$, and $A \neq X$, then $\exists\, a \in A$ and $x \in X - A$.

Since X is path-connected, $\exists\, \gamma: [0,1] \to X$, s.t. $\gamma(0) = a$, $\gamma(1) = x$.

$\emptyset \neq \gamma^{-1}(A) \subsetneq [0,1]$ is both open and closed, $\gamma^{-1}(A) = [0,1]$, contradiction.

Thus $A = X$, X is connected. $\square$

prop 2. If $X \subseteq \mathbb{R}^n$, then X is connected iff X is path-connected.

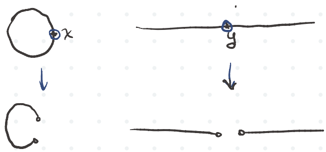
(proof is omitted here.)

Def. $\forall\, x \in X$, $P(x) := \{ y \in X \mid \exists\, \gamma: [0,1] \to X, \text{ s.t. } \gamma(0) = x, \gamma(1) = y \} \subseteq X$.

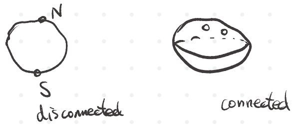
Then $P(x)$ is path-connected, and <u>maximal</u>: If $x \in A \subseteq X$ is path-connected, then $A \subseteq P(x)$.

We call $P(x)$ the <u>path-connected component</u> of X containing x.

Applications. ① $S^1 \not\cong \mathbb{R}^1$:



② S^1 and S^n $(n \geq 2)$ are not homeo.



$\mathbb{R}^1 \not\cong \mathbb{R}^n$ $(n \geq 2)$

$S^1 \not\cong 8$.

Lecture 3. Compactness.

Def. A space X is compact if every open cover of X has/admits finite open subcover.
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A cover is a collection $\{X_\alpha\}$ of subsets of X s.t. $X = \bigcup_\alpha X_\alpha$.

prop. Let $f: X \to Y$ be a surjective map. If X is compact, then so is Y.
(the image of compact space is compact).

proof. Let $\{V_\alpha\}$ be an open cover of Y. Then $U_\alpha = f^{-1}(V_\alpha)$ cover X.

Since X is compact, $X = U_1 \cup \cdots \cup U_n$, $U_1, \cdots, U_n \in \{U_\alpha\}$.

Thus $Y = f(X) = f(\bigcup_{i=1}^n U_i) = \bigcup_{i=1}^n f(U_i) = \bigcup_{i=1}^n V_i$; i.e. Y is compact. $\square$

Cor. Compactness is a topological property.

Example. $\mathbb{R}^1$ is not compact.

$$\mathbb{R}^1 = \bigcup_{i=1}^\infty (-i, i) \text{ doesn't admit finite subcover.}$$

Heine-Borel thm. (a,b) is not compact.

Fact. $[a,b]$ is a compact subset of $\mathbb{R}^1$, for any $a<b$.

Rmk. A subset $A \subseteq X$ is compact if every open cover $\{U_\alpha\}$ of A in X has a finite subcover $\{U_i \mid i=1,\cdots,n\}$, s.t. $A \subseteq \bigcup_{i=1}^n U_i$. $(n<\infty)$.

$$A = \bigcup_\alpha (V_\alpha \cap A), \quad V_\alpha \subseteq X \text{ open}.$$
$$= \bigcup_{i=1}^n (V_i \cap A) = \left(\bigcup_{i=1}^n V_i\right) \cap A$$
$$\Leftrightarrow A \subseteq \bigcup_{i=1}^n V_i.$$

prop. Closed subsets of a compact space are compact, as subspaces.

proof Let X be a compact space and let $A \subseteq X$ be closed. $X - A$ is open.
Given any open cover $\{U_\alpha\}$ of A in X, then $\{U_\alpha, X-A\}$ is an open cover of X.
Since X is compact, $X = (\bigcup_{i=1}^n U_i) \cup (X - A)$, which implies $A \subseteq \bigcup_{i=1}^n U_i$. A is compact. $\square$

« Counterexamples in Topology »

Prop. Compact subsets of a Hausdorff space are closed.

proof. Let X be a Hausdorff space and let $A \subseteq X$ be compact.

Need to show A is closed: $\overline{A} \subseteq A$.

We may assume that $\phi \neq A \subsetneq X$. Then $\exists\, x \in X \setminus A$.

Since X is Hausdorff, $\forall\, a \in A$, $\exists\, U_x$ and V_a in X st. $U_x \cap V_a = \phi$.

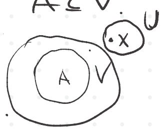
Let $V' = \bigcup_{a \in A} V_a$. Then V is an open cover of A.

Since A is compact, $A \subseteq V = \bigcup_{i=1}^{n} V_i$.

Let $U_1, \cdots, U_n$ be the corresponding nbhds of x such that $V_i \cap U_j = \phi$.

Form $U = \bigcap_{j=1}^{n} U_j$, then $x \in U$, $U \cap V = \phi$.

$A \subseteq V$. Thus every point out of A is not a limit point of A.



That is, every limit point of A lies in A: $\overline{A} \subseteq A$.

$\square$

Cor. Let $f: X \to Y$ be a surjective map with X compact and Y Hausdorff.

Then f is an open/closed map.

Moreover, if f is a bijective, then f is a homeomorphism.

proof. Exercise.

prop. Let X and Y be compact space, then $X \times Y$ is compact.

proof. Let $\{O_\alpha\}$ be an open cover of $X \times Y$. $\forall\, (x,y) \in X \times Y$,

$\exists\, (x,y) \in U_{xy} \times V_{xy} \subseteq O_\alpha$ for some α.

Fix x and let y vary, $\{U_{xy} \times V_{xy} \mid y \in Y\}$ is an open cover of $\{x\} \times Y$.

Since Y is compact, $Y \xrightarrow{\cong} \{x\} \times Y$, $\{x\} \times Y \subseteq \bigcup_{i=1}^{n} (U_{xy_i} \times V_{xy_i})$.

Let $U_x = \bigcap_{j=1}^{n} U_{xy_j}$ be the nbhd of x. Then $U_x \times V_{xy_1}, \cdots, U_x \times V_{xy_n}$ cover $\{x\} \times Y$.

Let x vary. Then $\{U_x\}$ is an open cover of X.

Since X is compact, $X = \bigcup_{i=1}^{m} U_i$, $U_i \in \{U_x\}$. Let $V_j = V_{xy_j}$.

Then $\{U_i \times V_j \mid i=1,\dots,m,\ j=1,\dots,n\}$ is an open cover of $X \times Y$.

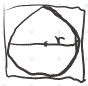
Thus $X \times Y$ is compact. $\square$

Rmk. If X_α are compact spaces, then $\prod_\alpha X_\alpha$ is compact.

Theorem. A subspace $X \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ X is bounded and closed.

proof. $\Leftarrow$: X is bounded: $\exists\, r > 0$ and $x \in X$, such that $X \subseteq B(x,r) \subseteq \mathbb{R}^n$.

$X \subseteq B(x,r) \subseteq [-r,r] \times \cdots \times [-r,r]$.



By the above Proposition and the fact that $[-r,r]$ is compact.

X is a closed subset of a compact space, X is compact.

$\Rightarrow$ Suppose $X \subseteq \mathbb{R}^n$ is compact.

$$\mathbb{R}^n = \bigcup_{r=1}^{\infty} B(0,r).$$

X is compact in $\mathbb{R}^n \Rightarrow X \subseteq \bigcup_{i=1}^{n} B(0,r_i) = B(0,r_n) \subseteq \mathbb{R}^n$.

Hence X is bounded.

Since $\mathbb{R}^n$ is Hausdorff, we derive that X is closed. $\square$

Cor. Let $f: X \to \mathbb{R}$ be a continuous function. If X is compact

then $\operatorname{Im} f = f(X) = [f(x_1), f(x_2)]$ for $x_1, x_2 \in X$.

proof. Since X is compact, $\operatorname{Im} f = f(X) \subseteq \mathbb{R}^1$ is compact. $\square$

Lebesgue number.

Let X be a metric space, the Lebesgue number for an open cover $\{A_\alpha\}$ of X is a number $\delta > 0$ s.t. any open ball $B(x,r)$ with $r < \delta$ is contained in some A_α.

Theorem: Every open cover of a compact metric space has a Lebesgue number. $\square$