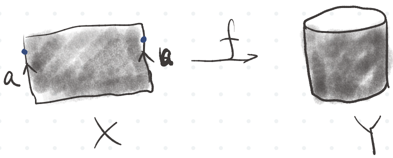
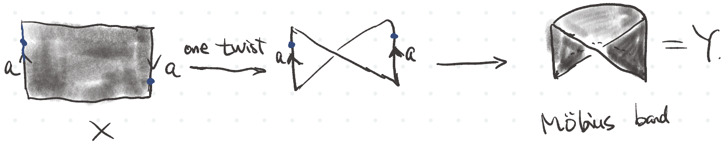
Lecture 4. Quotient Topology and Surfaces.
(identification topology)

Example:





Prop. Let $f: X \to Y$ be a surjective function from a topological space X onto a set Y. Then $\mathcal{T}_f = \{ V \subseteq Y \mid f^{-1}(V) \in \mathcal{T}_X \}$ is a topology on Y.

(Exercise)

· $\mathcal{T}_f$ is called the quotient/identification topology on Y.

In this setting, f is called a quotient map, it is an open map.

$$U = f^{-1}(V) \longrightarrow f(U) = f(f^{-1}(V)) = V$$

Lemma. Let $f: X \to Y$ be a quotient map.

(1) If $g: Y \to Z$ is a quotient map, then so is $g \circ f: X \to Z$.

(2) Universal property:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

Let $g: Y \to Z$ be any a function, then $g \circ f$ is continuous iff g is continuous.

$\square$

Observation. Let $f: X \to Y$ be a quotient map.

$\forall y \in Y$, $f^{-1}(y) = \{x \in X \mid f(x) = y\} \neq \phi$.

If $y \neq y'$, $f^{-1}(y) \cap f^{-1}(y') \neq \phi$.

$$X = \coprod_{y \in Y} f^{-1}(y)$$

Equivalence relation $\sim$: ① $x \sim y \Leftrightarrow y \sim x$
② $x \sim x$
③ $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Example: ① topological spaces
$X \sim Y \Leftrightarrow X \cong Y$
is an equivalence relation.
② matrices $A \sim B \Leftrightarrow A$ is similar to B.

$x \sim x' \Leftrightarrow x, x' \in f^{-1}(y) \Leftrightarrow f(x) = f(x')$.

Check the above relation is an equivalence relation.

$[x] = \{x' \in X \mid f(x') = f(x)\}$,

$X/\sim := \{[x] \mid x \in X\}$

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ {[x]} & X/\sim \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$\bar{f}([x]) = f(x)$.

$\bar{f}$ is well-defined: $[x] = [x']$, $f(x) = f(x')$, $\therefore \bar{f}([x]) = \bar{f}([x'])$.

$\bar{f}$ is unique.

- Every quotient map f can be factored as the composition $\bar{f}\pi$, $(f = \bar{f} \circ \pi)$ and $\bar{f}$ is a bijection.

- Theorem: Let $f: X \to Y$ be a quotient map and let $\bar{f}: X/\sim \to Y$ be as above. Then $\bar{f}: X/\sim \to Y$ is a homeomorphism.

proof: $\forall V \subseteq Y$ be open. $f^{-1}(V) \subseteq X$ is open, since f is a quotient map

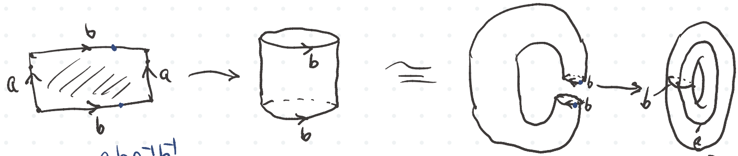
$f = \bar{f} \circ \pi$. $f^{-1}(V) = (\bar{f} \circ \pi)^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ is open and π is an open map imply that $\bar{f}^{-1}(V)$ is open.

Thus $V \subseteq Y$ is open iff $\bar{f}^{-1}(V)$ is open. $\square$

Examples.

Cut and paste

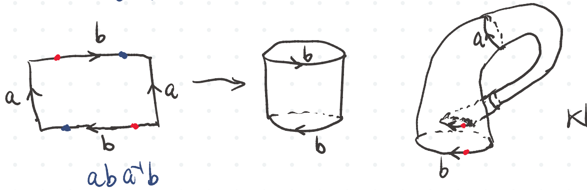
①



$aba^{-1}b^{-1}$

Torus T^2

②



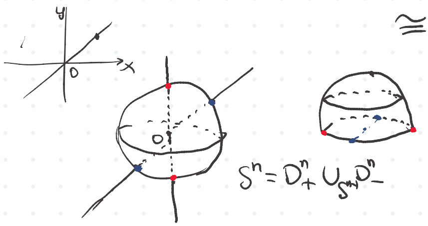
$aba^{-1}b$

Klein bottle K.

③ projective space P^n:

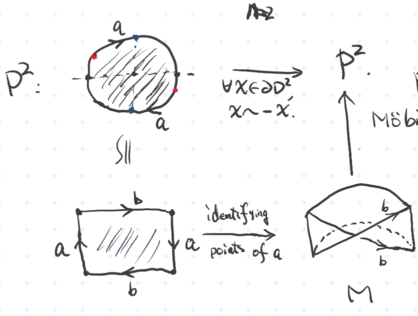
$$\mathbb{R}P^n = \mathbb{R}^{n+1}\setminus\{0\} \big/ x\sim kx,\ k\in\mathbb{R}\setminus\{0\} \cong S^n \big/ x\sim -x,\ \forall x\in S^n.$$

$$\cong D^n \big/ x\sim -x,\ \forall x\in \partial D^n = S^{n-1}.$$



$S^n = D^n_+ \cup_{S^{n-1}} D^n_-$

$n=2$

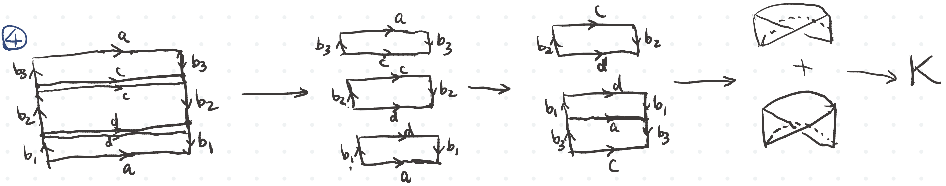


Lemma: $P^2 \cong M \cup_{S^1} D^2$

P^2 is the quotient space of the Möbius band with its boundry identified.

Cor. $M \cong P^2 \setminus \mathring{D}^2$

④



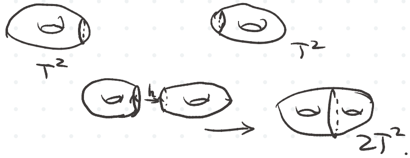
lemma. $K \cong M \cup_{S^1} M \cong (P^2\setminus\mathring{D}^2)\cup_{S^1}(P^2\setminus\mathring{D}^2) = P^2 \# P^2$.

Surfaces

A topological space S is called a <u>surface</u> if $\forall x \in S$, $\exists$ open nbhd U_x, and a homeomorphism $h_x : U_x \xrightarrow{\cong} \mathbb{R}^2$.

Examples T^2, P^2, $K = M \cup_{S^1} M$

$2T^2 = T^2 \# T^2$.



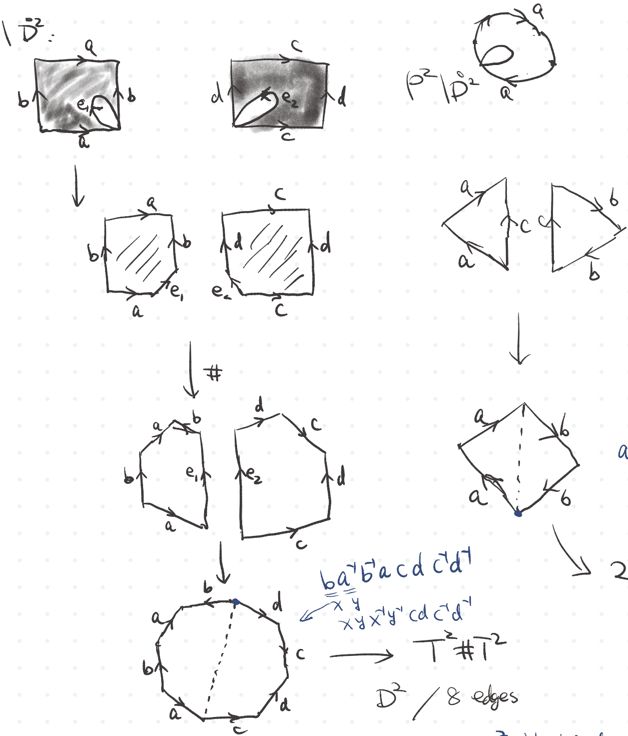
Connected sum of surfaces,

Given two surfaces S_1, S_2, their connected sum <u>$S_1 \# S_2$</u> is the quotient space

$$S_1 \# S_2 = (S_1 \setminus \mathring{D}_1) \cup_h (S_2 \setminus \mathring{D}_2), \quad h : S^1 = \partial D_1 \to S^1 = \partial D_2$$

h is a homeomorphism preserving orientation

Example. $T^2 \setminus \mathring{D}^2$:



$aabb = a^2 b^2$.

$2P^2 = P^2 \# P^2$

$ba^{-1}b^{-1}acdc^{-1}d^{-1}$

$xyx^{-1}y^{-1}cdc^{-1}d^{-1}$

$\to T^2 \# T^2$

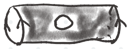
$D^2 / 8$ edges

Lemma. $nT^2 = T^2 \# \cdots \# T^2$ (n copies) is a quotient space of D^2 identified pairs of $4n$ edges

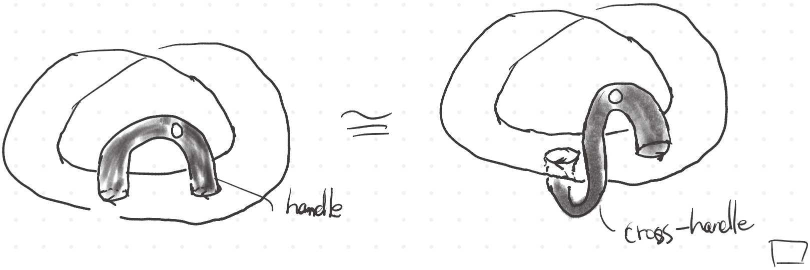
Lemma. $mP^2 = P^2 \# \cdots \# P^2$ (m copies) is quotient space of D^2 identified pairs of $2m$ edges.

Lemma. $P^2 \# T^2 \cong P^2 \# K$.

proof $\quad P^2 \# T^2 = (P^2 \setminus \mathring{D}^2) \cup_{S^1} (T^2 \setminus \mathring{D}^2)$

$= M \cup_{S^1}$ ()

$P^2 \# K = M \cup_{S^1}$ ()

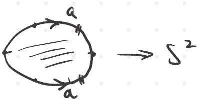


closed surface = compact surface

Theorem (Classification Theorem of Surfaces)

(1) Every compact connected surface is homeomorphic to one of the followings:

(i) 2-Sphere S^2



(ii) $nT^2 = T^2 \# \cdots \# T^2$ (n copies)

(iii) $mP^2 = P^2 \# \cdots \# P^2$ (m copies)

(2) Any two surfaces above are not homeomorphic. (cannot be proved yet)

Proof (Sketch): (Massey, A basic course in algebraic Topology, Chapter I)

Fact: Every compact connected surface is the quotient space of a single disk with pairs of edges indentified in its boundry; moreover, all vertices of edges will be identified to one vertex.

· Firstly we identify the two vertices of all edges to get pairs of loops.

It follows that every compact connected surface is the quotient space of a sphere with interiors of (pairs of) disk removed.

① If a pair of loops have same orientations, the identification is equivalent to attach a handle $S^1 \times [0,1]$ to them.

(Equivalently, the connected sum with T^2)

② If a pair of loops have different orientations, the identification is equivalent to attach a cross-handle to them. (equivalently, the connected sum with K)

③ If there exists a loop identified with itself by identifying its antipole points then the identification is equivalent to attach a $\underbrace{\text{Cross-cap}}_{\text{Möbius band}}$ to them (equivalently, the connected sum with P^2)

Thus. ① If all loops have consistent/same direction $\Rightarrow S^2 \# nT^2 \cong \underline{\underline{nT^2}}$.

② If there are pairs of loops having different directions, $S^2 \# mK \# nT^2 \cong \begin{cases} mK \cong 2mP^2 & n=0 \\ s\cdot P^2 & n>0 \end{cases}$

Lemma: $K = P^2 \# P^2$.

③ If there exists one loop identified with itself, $P^2 \# nT^2 \# mK \cong \underline{\underline{t\cdot P^2}}$.

Therefore (1) is proved. $\square$

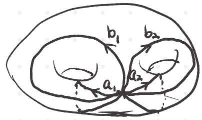
Standard representations of surfaces.

(i) S^2: aa^{-1}

(ii) nT^2: $a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1} \xlongequal{[x,y]=xyx^{-1}y^{-1}} [a_1,b_1]\cdots[a_n,b_n]$

(iii) mP^2: $a_1^2\cdots a_m^2$

$n=2$.



Euler Characteristics of Surfaces.

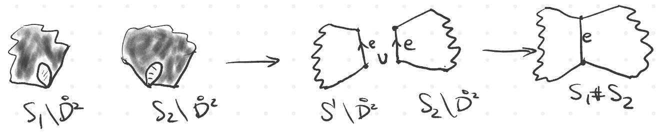
Recall: $\chi(S^2)=\chi(P)=2$.

Prop. Let S_1 and S_2 be two compact connected surfaces. There holds:

$$\chi(S_1\# S_2)=\chi(S_1)+\chi(S_2)-2.$$

- By induction, $\chi(S_1\#\cdots\# S_n)=\chi(S_1)+\cdots+\chi(S_n)-2(n-1)$.

proof.



$$v(S_1\# S_2)=v(S_1)+v(S_2)-1$$
$$f(S_1\# S_2)=f(S_1)+f(S_2)-1$$
$$e(S_1\# S_2)=e(S_1)+e(S_2)$$
$$\Rightarrow \chi(S_1\# S_2)=v(S_1\# S_2)-e(S_1\# S_2)+f(S_1\# S_2)$$
$$=v(S_1)+v(S_2)-1-e(S_1)-e(S_2)+f(S_1)+f(S_2)-1$$
$$=\chi(S_1)+\chi(S_2)-2. \quad \square$$

$S^2=D^2/S^1$

S^2: aa^{-1}

$\chi(S^2)=v-e+f=1-0+1=2$

nT^2: $a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}$

$\chi(nT^2)=v-e+f=1-2n+1=2-2n$

mP^2: $a_1^2a_2^2\cdots a_m^2$ $\quad\chi(mP^2)=v-e+f=1-m+1=2-m$

Exercise.
Check the formulas on the left by the formula:
$\chi(S_1\# S_2)=\chi(S_1)+\chi(S_2)-2$.

- $\chi(S)\le 2$.

Theorem. Any two surfaces are homeomophic iff

(i) they are both orientable or both non-orientable

(ii) they have equal Euler characteristics.

(proofs will be given later.)