Lecture 05. Fundamental Groups / Poincaré groups.

Groups and Homomorphisms of group.

A group G is a set G with a binary operation $(\mu, \cdot)$, such ~~that~~
$(G, \cdot)$ $(\mu, \cdot : G \times G \longrightarrow G)$

(i) associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $\forall x, y, z \in G$.

(ii) unit e: $x \cdot e = x = e \cdot x$, $\forall x \in G$.

(iii) inverse: $\forall x \in G, \exists y \in G$, s.t. $x \cdot y = e = y \cdot x$, $y = x^{-1}$.

Example: $G = (\mathbb{Z}, +)$, $k^{-1} = -k$, $k \in G$, $e = 0$.

$G = (\mathbb{R}, +)$, $(\mathbb{R}, \cdot)$; $G = (\mathbb{Z}/m\mathbb{Z}, +)$, $\mathbb{Z}/m\mathbb{Z} = \{\bar{0}, \bar{1}, \cdots, \overline{m-1}\}$

$\bar{k} = k + m\mathbb{Z}$: $\bar{k}_1 = \bar{k}_2 \Leftrightarrow \underbrace{k_1 - k_2 = t \cdot m}_{\in m\mathbb{Z}}$.

A function $\varphi : (G_1, \cdot) \longrightarrow (G_2, *)$ is a homomorphism of groups if
$$\varphi(xx') = \varphi(x)\varphi(x'), \quad \varphi(e_1) = e_2.$$

Homotopic maps.

Two maps $f, g : X \longrightarrow Y$ are homotopic if there is a map
$$F : X \times I \longrightarrow Y, \quad I = [0,1] \quad \text{such that} \quad F(x,0) = f(x),\ F(x,1) = g(x).$$
$F(x,t) =$

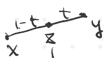
F is called a homotopy from f to g, denoted by $F : f \simeq g : X \longrightarrow Y$.
or $f \overset{F}{\simeq} g : X \longrightarrow Y$.

Define $F_t : X \longrightarrow Y$, $F_t(x) = F(x,t)$, is a continuous map.
$F_0 = f$, $F_1 = g$.

Example: Any two maps $f, g : X \longrightarrow \mathbb{R}^n$ are homotopic:
$$F : X \times I \longrightarrow \mathbb{R}^n, \quad F(x,t) = (1-t) \cdot f(x) + t \cdot g(x).$$
line segment homotopy.

More general, any two maps $f, g : X \longrightarrow C \subseteq \mathbb{R}^n$ with C convex are homotopic.

C is convex means that $\forall x, y \in C$, $(1-t)x + ty \in C$, $\forall t \in [0,1]$.



$z = (1-t) \cdot x + t \cdot y$

prop. The homotopy relation $\simeq$ on the set $C(X,Y)$ of maps from X to Y is an equivalence relation.

proof. Exercise.

$[f] := \{g : X \to Y \mid f \simeq g\} \ni f$, $\langle X, Y\rangle := C(X,Y)/\simeq$

homotopy relative to subsets.

Let $A \subseteq X$ be a subset (including the case $A = \phi$), two maps $f, g : X \to Y$ are homotopic relative to A (denoted by $f \simeq g$ rel A) if there exists a map

$F : X \times I \to Y$, such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $F(a,t) = f(a) = g(a)$ $\forall a \in A, t \in I$.

$F_t : X \to Y$, $F_t(x) = F(x,t)$, $F_t|_A = f|_A = g|_A$.

If $A = \phi$ $\rightsquigarrow$ absolute homotopy

prop. The homotopy relative to subsets on the set $C_A(X,Y)$ of continuous maps from X to Y which are "constant" on A, is an equivalence relation.

- $\langle X, Y\rangle_A := C_A(X,Y)/\simeq_{\text{rel} A}$

$A = \phi \longrightarrow$ absolute homotopy.

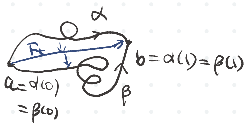
$A = \{*\} \longrightarrow$ based homotopy : $[X, Y] = \langle X, Y\rangle_*$

$A = \{x_1, x_2\} \longrightarrow$ ...

Homotopy of Path / Construction of the fundamental group $\pi_1(X, x_0)$.

$\alpha, \beta : I \to X$ are path-homotopic if $\alpha \overset{F}{\simeq} \beta$ rel $\{0,1\} = \partial I$.

$[\alpha] = \{\beta : I \to X \mid \beta \simeq \alpha \text{ rel } \partial I\}$



$\alpha_0 \overset{\alpha_t}{\simeq} \alpha_1$: $\alpha_0 \simeq \alpha_1$ rel ∂I.

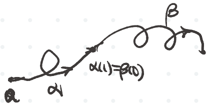
$\alpha_t : I \to X$, $\alpha_t(0) = a$, $\alpha_t(1) = b$, $\forall t \in I$.

$\alpha_t = F(\cdot, t)$

$F : I \times I \to X$

Operation on paths

Given two paths $\alpha, \beta : I \to X$, $\alpha(1) = \beta(0)$, define



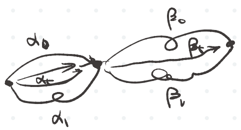
$\alpha \cdot \beta : I \longrightarrow X$ is a path defined by

$$(\alpha\cdot\beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 ; \\ \beta(2t-1) & 1/2 \le t \le 1 . \end{cases}$$

Recall: $f : X = X_1 \cup X_2 \longrightarrow Y$, if $f|X_1$ and $f|X_2$ are continuous,

$X_1, X_2 \subseteq X$ are closed, then $f = f_1 \cup f_2 : X \to Y$ is continuous.

Lemma. Let $\alpha_0 \overset{\alpha_t}{\simeq} \alpha_1 : I \to X$, and let $\beta_0 \overset{\beta_t}{\simeq} \beta_1 : I \to X$, $\alpha_t(1) = \beta_t(0)$.



Then $\alpha_0 \cdot \beta_0 \overset{\alpha_t \cdot \beta_t}{\simeq} \alpha_1 \cdot \beta_1$. $\square$

Define $[\alpha]\cdot[\beta] := [\alpha\cdot\beta]$ is well-defined when composable/multipliable.

Loop: $\alpha : I \to X$ is a loop if $\alpha(0) = \alpha(1)$.

$[\alpha]\cdot[\beta] := [\alpha\cdot\beta]$

$$(*) \quad (\alpha\cdot\beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le 1/2 ; \\ \beta(2t-1) & 1/2 \le t \le 1 \end{cases}$$

Theorem (Poincaré, 1890')

The set $\pi_1(X, x_0)$ of based loops in X at x_0 is a group.

$\pi_1(X, x_0) = [I, X]_*$, $[\gamma] \in \pi_1(X, x_0) \Leftrightarrow \gamma : I \to X$, $\gamma(0) = \gamma(1) = x_0$

$[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$ is given by $(*)$.

Lemma: Let $\gamma: I \to X$ be a path, $\gamma(0)=a$, $\gamma(1)=b$. Let $\rho: I \to I$ be a map such that $\rho(0)=0$, $\rho(1)=1$, then $\gamma\circ\rho \simeq \gamma$: $[\gamma\circ\rho]=[\gamma]$.

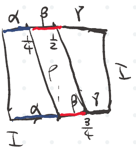
proof. $\rho: I \to I$, $\rho \simeq id_I$: $\underline{(1-t)\rho(s)+t\cdot s}$

$\gamma_t(s) = \gamma((1-t)\rho(s)+t\cdot s)$, $0\le s, t\le 1$, gives a homotopy between $\gamma\circ\rho$ and γ, rel ∂I. $\square$

proof of Poincaré theorem.

(1) Associativity. $([\alpha]\cdot[\beta])\cdot[\gamma] = [\alpha]\cdot([\beta]\cdot[\gamma]) \Leftrightarrow (\alpha\cdot\beta)\cdot\gamma \simeq \alpha\cdot(\beta\cdot\gamma)$.

$$(\alpha\cdot\beta)\cdot\gamma\,(s) = \begin{cases} \alpha(4s), & 0\le s\le 1/4 \\ \beta(4s-1), & 1/4\le s\le 1/2 \\ \gamma(2s-1) & 1/2\le s\le 1 \end{cases}$$



take $\rho(s) = \begin{cases} s/2 & 0\le s\le 1/2 \\ s-\frac{1}{4} & 0\le s\le 3/4 \\ 2s-1 & \frac{3}{4}\le s\le 1 \end{cases}$

$\rho: I \to I$

$$\alpha\cdot(\beta\cdot\gamma)(s) = \begin{cases} \alpha(2s) & 0\le s\le 1/2 \\ \beta(4s-2) & 1/2\le s\le 3/4 \\ \gamma(4s-3) & 3/4\le s\le 1 \end{cases}$$

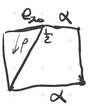
$$=[(\alpha\cdot\beta)\cdot\gamma](\rho(s))$$

Thus by the Lemma, $[(\alpha\cdot\beta)\cdot\gamma]\circ\rho = \alpha\cdot(\beta\cdot\gamma) \simeq (\alpha\cdot\beta)\cdot\gamma$.

(2) Unit: $e_{x_0}: I \to X$, $e_{x_0}(t)\equiv x_0$, $\forall t\in I$.

$[e_{x_0}]\cdot[\alpha] = [\alpha] = [\alpha]\cdot[e_{x_0}] \Leftrightarrow e_{x_0}\cdot\alpha \simeq \alpha \simeq \alpha\cdot e_{x_0}$

$$(e_{x_0}\cdot\alpha)(s) = \begin{cases} e_{x_0}(2s-1)\equiv x_0 & 0\le s\le 1/2 \\ \alpha(2s-1) & 1/2\le s\le 1 \end{cases}$$



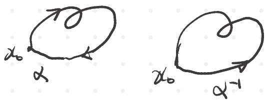
$$\rho(s) = \begin{cases} 0 & 0\le s\le 1/2 \\ 2s-1 & 1/2\le s\le 1 \end{cases}$$

Then by the Lemma, $e_{x_0}\cdot\alpha \simeq \alpha$.

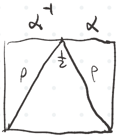
(3) $[\alpha]^{-1} = [\alpha^{-1}] \iff [\alpha^{-1}]\cdot[\alpha] = [e_{x_0}] = [\alpha]\cdot[\alpha^{-1}] \iff \underline{\alpha^{-1}\cdot\alpha \simeq e_{x_0} \simeq \alpha\cdot\alpha^{-1}}.$

Given $\alpha : I \longrightarrow X$, $\alpha^{-1} : I \to X$, $\alpha^{-1}(t) = \alpha(1-t)$.

$$(\alpha^{-1}\cdot\alpha)(s) = \begin{cases} \alpha^{-1}(2s) = \alpha(1-2s) & 0 \le s \le 1/2 \\ \alpha(2s-1) & 1/2 \le s \le 1. \end{cases}$$



$\rho(s) = ?$



Exercise: Find $\rho : I \to I$ such that

$\alpha^{-1}\alpha \simeq (\alpha^{-1}\cdot\alpha)\circ\rho = e_{x_0}$.

$\square$