Lecture 6. Fundamental Groups II

Review: $\pi_1(X, x_0) =$ the set of homotopy classes of loops based at $x_0 / \simeq$

loop $\gamma: I \to X$, $\gamma(\partial I) = x_0$

$$\begin{array}{ccc} \gamma: (I, \partial I) \longrightarrow (X, x_0) & \Leftrightarrow & \gamma: (S^1, s_0) \longrightarrow (X, x_0) \\ \downarrow & & \\ I/\partial I = (S^1, s_0) \longrightarrow (X, x_0) & & \pi_1(X, x_0) = [S^1, s_0; X, x_0]_* \end{array}$$

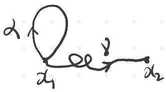
$\triangle$ $\pi_1(X, x_0) = [S^1, s_0; X, x_0]$ is a group under the multiplication of loops.

Independence of the choices of base points.

prop. Let X be a path-connected space. Then $\pi_1(X, x_1) \cong \pi_1(X, x_2)$.

proof. Since X is path-connected, there exist a path $\gamma: I \to X$ s.t. $\gamma(0) = x_1, \gamma(1) = x_2$.

$$\begin{aligned} \phi_\gamma: \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_2) \\ [\alpha] &\longmapsto [\gamma^{-1} \cdot \alpha \cdot \gamma] \end{aligned}$$



ϕ_γ is well-defined: if $\alpha_0 \overset{\alpha_t}{\simeq} \alpha_1$, then $\gamma^{-1} \cdot \alpha_0 \cdot \gamma \overset{\gamma^{-1} \cdot \alpha_t \cdot \gamma}{\simeq} \gamma^{-1} \cdot \alpha_1 \cdot \gamma$.

$\psi_\gamma: \pi_1(X, x_2) \longrightarrow \pi_1(X, x_1)$

$\psi_\gamma([\beta]) = [\gamma \cdot \beta \cdot \gamma^{-1}]$. Similarly, ψ_γ is well-defined.

Check that $\phi_\gamma \circ \psi_\gamma = id$, $\psi_\gamma \circ \phi_\gamma = id$.

ϕ_γ is a homomorphism of groups:

$$\begin{aligned} \phi_\gamma([\alpha][\alpha']) = \phi_\gamma([\alpha \cdot \alpha']) &= [\gamma^{-1} \cdot (\alpha \cdot \alpha') \cdot \gamma] \\ &= [(\gamma^{-1} \cdot \alpha \cdot \gamma)(\gamma^{-1} \cdot \alpha' \cdot \gamma)] \\ &= [\gamma^{-1} \cdot \alpha \cdot \gamma] \cdot [\gamma^{-1} \cdot \alpha' \cdot \gamma] \\ &= \phi_\gamma([\alpha]) \cdot \phi_\gamma([\alpha']). \end{aligned}$$

Similarly, ψ_γ is a homomorphism.

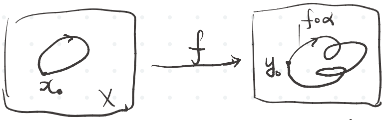
Thus $\phi_\gamma: \pi_1(X, x_1) \to \pi_1(X, x_2)$ is an isomorphism. $\square$

- If X is path-connected, we usually denote $\pi_1(X) = \pi_1(X, x)$, $\forall x \in X$.

Naturality. Every map $f:(X,x_0)\to(Y,y_0)$ induces a homomorphism

$$f_\#:\pi_1(X,x_0)\longrightarrow\pi_1(Y,y_0).$$

proof. Define $f_\#([\alpha])=[f\circ\alpha]$. Clearly $f_\#$ is well-defined.



$$f_\#([\alpha]\cdot[\alpha'])=f_\#([\alpha\cdot\alpha'])$$
$$=[f\circ(\alpha\cdot\alpha')]$$

$$f\circ(\alpha\cdot\alpha')(t)=\begin{cases}(f\circ\alpha)(2t) & 0\le t\le 1/2\\ (f\circ\alpha')(2t-1) & 0\le t\le 1/2\end{cases}=(f\circ\alpha)\cdot(f\circ\alpha')(t)$$

$\therefore\ f\circ(\alpha\cdot\alpha')=(f\circ\alpha)\cdot(f\circ\alpha')$ Thus $f_\#$ is a homomorphism. $\square$

- prop. $(g\circ f)_\#=g_\#\circ f_\#$, $(\mathrm{id}_X)_\#=\mathrm{id}_{\pi_1(X)}$. $\square$

Cor. The ~~F~~ fundamental group $\pi_1(X,x_0)$ is a topological invariant.

If $X\cong Y$, then $\pi_1(X,x_0)\overset{\cong}{\to}\pi_1(Y,y_0)$.

$g\circ f=\mathrm{id}_X$, $f\circ g=\mathrm{id}_Y\Rightarrow g_\#\circ \underline{\underline{f_\#}}=\mathrm{id}$, $\underline{\underline{f_\#}}\circ g_\#=\mathrm{id}$.

(左单右满: $A\xrightarrow{\varphi}B\xrightarrow{\phi}C$

if $\phi\varphi$ is surjective, then ϕ is surjective.

if $\phi\varphi$ is injective, then φ is injective)

$\Rightarrow f_\#$ is injective and surjective

$\Rightarrow f_\#$ is an isomorphism. $\square$

★ **Theorem.** $\pi_1(S^1)\cong\mathbb{Z}$. $\phi:\mathbb{Z}\longrightarrow\pi_1(S^1)$, $\phi(n)=[\gamma_n]$, $\gamma_n(t)=e^{i2\pi nt}$.

- product spaces $\pi_1(X\times Y,(x_0,y_0))\cong\pi_1(X,x_0)\oplus\pi_1(Y,y_0)$.

proof. $S^1=I/\partial I\xrightarrow{\alpha}X\times Y$

$\alpha=(\alpha_1,\alpha_2)$, $\alpha_1=p_1\circ\alpha$
$\alpha_2=p_2\circ\alpha$

$p_1:X\times Y\to X$, $p_2:X\times Y\to Y$.

$\varphi:\pi_1(X\times Y,(x_0,y_0))\longrightarrow\pi_1(X,x_0)\oplus\pi_1(Y,y_0)$
$[\alpha]\longmapsto[\alpha_1=p_1\circ\alpha,\ \alpha_2=p_2\circ\alpha]$.

φ is injective. $\mathrm{Ker}\,\varphi=\{[\alpha]\mid \alpha_1\simeq e_{x_0},\ \alpha_2\simeq e_{y_0}\}$
$=\{[(\alpha_1,\alpha_2)]=[e_{x_0},e_{y_0}]\}=0$

φ is surjective: $\beta\in\pi_1(X,x_0)$, $\gamma\in\pi_1(Y,y_0)$
$\alpha=(\beta,\gamma)\longmapsto(\beta,\gamma)$. $\square$

Example. $T^2 \cong S^1 \times S^1$ $\quad \pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1)$

 $\cong S^1 \times S^1$

$\cong \mathbb{Z} \oplus \mathbb{Z} \langle \alpha, \beta \rangle$

$\alpha = [a], \beta = [b]$.

· Rmk. By induction, $\pi_1(\prod_{i=1}^n X_i) \cong \bigoplus_{i=1}^n \pi_1(X_i)$.

More general, $\pi_1\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_1(X_\alpha)$.

Homotopic Spaces.

Def. Two $\underset{\text{(based)}}{}$spaces X and Y are said to be homotopic or be homotopy equivalent or have the same homotopy type if there exist $\overset{\text{based}}{}$maps $f: X \to Y$, $g: Y \to X$ such that $g \circ f \overset{F}{\simeq} \mathrm{id}_X$ (based), $f \circ g \overset{G}{\simeq} \mathrm{id}_Y$ (based). Notation: $X \simeq Y$.

By definition, homeomorphic spaces are homotopy equivalent.

$X \cong Y \Rightarrow X \simeq Y$.

Lemma. (i) The $X \simeq Y$ relation is an equivalent relationship in the set

$Top = \{\text{topological spaces}\}$.

$Top_* = \{\text{based topological spaces}\}$

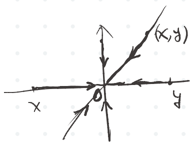
(ii) The $X \simeq Y$ relation is an equivalent relationship in Top_*.

Examples. ① $\mathbb{R}^n \simeq \{*\}$. $\quad \mathbb{R}^n \xrightarrow{r} \{0\}$, $\{0\} \xrightarrow{i} \mathbb{R}^n$

$r \circ i = \mathrm{id} : \{0\} \to \{0\}$.

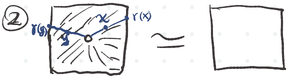
$i \circ r \simeq \mathrm{id}_{\mathbb{R}^n}$. ✓ $\mathbb{R}^n \xrightarrow{r} \{0\} \xrightarrow{i} \mathbb{R}^n$

Recall that any two maps into $\mathbb{R}^n$ (a convex space) are homotopic.



Thus $i \circ r \simeq \mathrm{id}_{\mathbb{R}^n}$

②



$$D^2\setminus\{0\} \xrightarrow{r} \partial D^2 = S^1 \xrightarrow{i} D^2\setminus\{0\} \quad \therefore\ D^2\setminus\{0\} \simeq S^1.$$

$$x \longmapsto r(x) = \frac{x}{\|x\|}$$

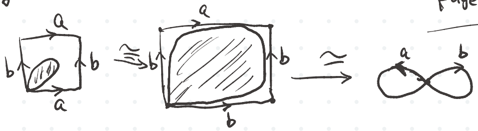
$r\circ i = id.\quad i\circ r \simeq id.$

③  $\simeq S^1$

④ 

$= T^2\setminus\{*\} \simeq S^1\vee S^1$: 

Page 22-23 Exercise 11 (b).



Def. Let $i: A \hookrightarrow X$ be the inclusion of subspace. If there exists a map $r: X \to A$ such that $r\circ i = id_A$, $i\circ r \simeq id_X$,

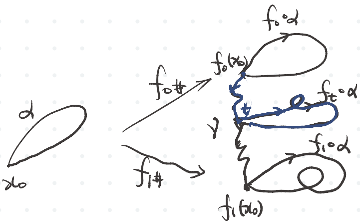
then we say ~~that~~ A is a deformation retraction of X.

The map $r: X \to A$ is called a retract.

prop. If $f_0 \overset{f_t}{\simeq} f_1 : X \to Y$, then $f_{0\#} = \Phi_\gamma \circ f_{1\#}$, $\gamma(t) = f_t(x_0)$

$$\begin{tikzcd} & \pi_1(Y, f_0(x_0)) \\ \pi_1(X, x_0) \arrow[ur, "f_{0\#}"] \arrow[dr, "f_{1\#}"'] & \\ & \pi_1(Y, f_1(x_0)) \arrow[uu, "\Phi_\gamma"', "\cong"] \end{tikzcd}$$

proof.



$f_{0\#}([\alpha]) = [f_0\circ\alpha]$

$\Phi_\gamma \circ f_{1\#}([\alpha]) = [\gamma\cdot(f_1\circ\alpha)\cdot\gamma^{-1}]$

Need to show $f_0\circ\alpha \simeq \gamma\cdot(f_1\circ\alpha)\cdot\gamma^{-1}$.

$\gamma_t\cdot(f_t\circ\alpha)\cdot\gamma_t^{-1}$

Let γ_t be the restriction of γ on $[0,t]$, $\gamma_t(s) = \gamma(ts)$. $\square$

$$[0,1] \xrightarrow{\cdot t} [0,t] \xrightarrow{\gamma} Y$$

Cor. ① If $f_t : f_0 \simeq f_1 : X \to Y$ is a based homotopy, $f_t(x_0) = y_0$,
then $f_{0\#} = f_{1\#}$. (clear)

② If X and Y are homotopy equivalent, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

proof. $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad f \circ g \simeq id_Y$, $g \circ f \simeq id_X$

By the proposition above,

$f_\# \circ g_\# = (f \circ g)_\# = \phi_\gamma$ for some path γ.

Since ϕ_γ is an isomorphism, $f_\#$ is surjective.

$g_\# \circ f_\# = (g \circ f)_\# = \phi_{\gamma_1}$ is an isomorphism $\Rightarrow f_\#$ is injective.

Thus $f_\#$ is an isomorphism. $\square$

presentation of groups by generators and relations.

群表现 $\quad G = \langle X \mid R \rangle = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle \quad r_1 = e, \cdots r_m = e$

Examples. $\mathbb{Z} = \langle x \mid \phi \rangle$, $\mathbb{Z} \oplus \mathbb{Z} = \langle x, y \mid xyx^{-1}y^{-1} = e \rangle$
$\Leftrightarrow xy = yx$

$\mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n \rangle$

Free products of groups.

containing the formal inverse.

Let X be a set, $F(X) = \langle \{x_1 \cdots x_m \mid x_i \in X\} \rangle$, free group generated by X.

$Y = X \sqcup X^{-1}$ $\quad (x_1 \cdots x_m) \cdot (y_1, \cdots, y_n) = x_1 \cdots x_m y_1 \cdots y_n$.

$\mathbb{Z} = F(x)$, $F_2 = \langle x_1, x_2 \rangle \xrightarrow{P} \langle x_1, x_2 \mid x_1 x_2 = x_2 x_1 \rangle = \mathbb{Z} \oplus \mathbb{Z}$

$F_n = \langle x_1, \cdots, x_n \rangle$

Fact. Every group is a quotient group of some free group.

$G_1 * G_2 := F(G_1 \sqcup G_2)$ is called the free product of G_1 and G_2

Van-Kampen Theorem. Let $X = X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$, $X_1, X_2, X_0 = X_1 \cap X_2$ are path-connected open subsets of X.

Seifert–van Kampen Theorem.

Then there is an isomorphism

$$\pi_1(X, x_0) \cong \pi_1(X_1, x_0) * \pi_1(X_2, x_0) / N$$

$$N = \langle\{ i_{1\#}(\alpha)\, i_{2\#}(\alpha)^{-1} \mid \alpha \in \pi_1(X_1 \cap X_2, x_0) \}\rangle$$

$$i_1 : X_1 \cap X_2 \longrightarrow X_1, \quad i_2 : X_1 \cap X_2 \longrightarrow X_2.$$

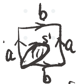
- If $\pi_1(X_1, x_0) \cong \langle A_1 \mid R_1 \rangle$, $\pi_1(X_2, x_0) \cong \langle A_2 \mid R_2 \rangle$, $\pi_1(X_1 \cap X_2, x_0) \cong \langle A_0 \mid R_0 \rangle$

  then $\pi_1(X, x_0) \cong \langle A_1 \amalg A_2 \mid R_1 \amalg R_2 \amalg R \rangle$,

  $R = \{ i_{1\#}(\alpha)\, i_{2\#}(\alpha)^{-1} \mid \alpha \in A_0 \}$.

Applications. $\pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ $\quad$ $\underline{n}T^2$ $\quad$ n is called the genus of nT^2.
亏格

$$\pi_1(T^2 \# T^2) \cong \pi_1(X_1) * \pi_1(X_2) / N$$
$$= \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$



$i_1[S^1]$

$[S^1] \mapsto [aba^{-1}b^{-1}]$

$[S^1] \overset{i_2}{\mapsto} [cdc^{-1}d^{-1}]$

$X_1 =$   $= X_2$ $\quad$ $X_1 \cap X_2 \simeq S^1 \xrightarrow{i_1} X_1$

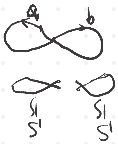
$\simeq$ $\quad$ $\simeq$

$S^1 \vee S^1$ (a, b) $\quad$ $S^1 \vee S^1$ (c, d)

$\pi_1(X_1) \cong \pi_1(S^1 \vee S^1) \cong F_2(a, b)$

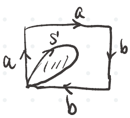
$\pi_2(X_2) \cong \pi_1(S^1 \vee S^1) \cong F_2(c, d)$

$\therefore \langle a, b, c, d \mid aba^{-1}b^{-1} \rangle \leq \pi_1(X_1)$



$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) / N{=}0$$
$$\cong \pi_1(S^1) * \pi_1(S^1)$$
$$= F_2$$

$$\pi_1(K) \cong \pi_1(P^2 \# P^2) \cong \langle a, b \mid a^2 b^2 \rangle$$



$i_{1\#}[S^1]\, i_{2\#}[S^1] = a^2 b^2$

$\underline{\pi_1(P^2)}$

In general nT^2: $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$

mP^2: $a_1^2 \cdots a_m^2$

Theorem. $\pi_1(nT^2) \cong \langle a_1, b_1, \cdots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$

$$\pi_1(mP^2) \cong \langle a_1, \cdots, a_m \mid a_1^2 \cdots a_m^2 \rangle$$

proof. Exercise. $\square$

Reference. Massey. A basic course in algebraic topology.

Hatcher. Algebraic Topology.

$X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

$\Leftrightarrow \pi_1(X) \not\cong \pi_1(Y) \Rightarrow X \not\cong Y$.

Cor. mP^2 and nT^2 are not homeomorphic.

mP^2 and $m'P^2$ are not — —

nT^2 and $n'T^2$ are not - - - - .

This completes the proof of part (2) of the classification theorem of surfaces. $\square$

Other Applications of fundamental groups.

We will not introduce covering spaces.

Next lecture Singular homology groups.