Lecture 07. Fundamental Groups III

Review: ① $\pi_1(X, x_0) = [I, \partial I; X, x_0]$

$I = [0,1]$ $\quad = [S^1, s_0; X, x_0] = [S^1, X]$ is a group under the multiplication

$$[\alpha]\cdot[\beta] = [\alpha\cdot\beta].$$

② $\pi_1(X, x_0)$ is independent of the choices of base points x_0.

$$\phi_\gamma : \pi_1(X, x_0) \cong \pi_1(X, x_1) \text{ if } \exists\, \gamma : I \to X,\ \gamma(0) = x_0,\ \gamma(1) = x_1.$$

$$[\alpha] \longmapsto [\gamma^{-1}\alpha\cdot\gamma] = [\gamma]^{-1}\cdot[\alpha]\cdot[\gamma]$$

③ $\pi_1(X, x_0) \xrightarrow{f_\#} \pi_1(Y, y_0)$: $(gf)_\# = g_\# \circ f_\#$; $(id_X)_\# = id_{\pi_1(X)}$

③ $\pi_1(X, x_0)$ is a homotopy invariant. $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$.

$(\Leftrightarrow$ "$\pi_1(X) \not\cong \pi_1(Y) \Rightarrow X \not\simeq Y$")

④ Van Kampen Theorem

$X = X_1 \cup X_2$, $X_1 \cap X_2$, $X_1, X_2 \subseteq X$ are path-connected open subsets.

$x_0 \in X_1 \cap X_2$, then there is an isomorphism.

$$\pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2) / N,$$

$$N = \langle \{ i_{1\#}(\alpha)\, i_{2\#}(\alpha)^{-1} \mid \forall \alpha \in \pi_1(X_1 \cap X_2) \} \rangle$$

$$\begin{tikzcd}
 & \pi_1(X_1) \arrow[dr, "i'_{1\#}"] & \\
\pi_1(X_1 \cap X_2) \arrow[ur, "i_{1\#}"] \arrow[dr, "i_{2\#}"'] & & \pi_1(X) \\
 & \pi_2(X_2) \arrow[ur, "i'_{2\#}"'] &
\end{tikzcd}$$

Examples. ① $\pi_1(S^1) \cong \mathbb{Z}$.

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

$S^1 \vee S^1 = (S^1, s_1) \amalg (S^2, s_2) / s_1 \sim s_2$

$$\pi_1(\vee_{i=1}^{n} S^1) \cong \pi_1(\vee_{i=1}^{n-1} S^1) * \pi_1(S^1) \cong *_{i=1}^{n} \mathbb{Z}.$$

$\cong F_n$, the free group generated by n elements.

true for $n = \infty$.

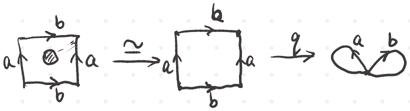
$X_1 =$  $\simeq S^1$

$X_2 =$  $\simeq S^1$

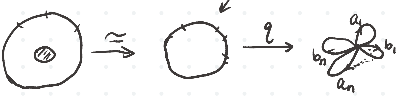
$X_1 \cap X_2 \simeq x_0 \simeq \{*\}$

$\pi_1(X_1 \cap X_2) \cong \pi_1(*) = 0$

② $T^2 \setminus \mathring{D}^2 \simeq S^1_a \vee S^1_b$



nT^2 is represented by $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$



$= \bigvee_{i=1}^{n} (S^1_{a_i} \vee S^1_{b_i})$, $S^1_{a_i} = S^1_{b_i} \cong S^1$.

$= \bigvee_{i=1}^{2n} S^1$.

Thus $nT^2 \setminus \mathring{D}^2 \simeq \bigvee_{i=1}^{2n} S^1 = \bigvee_{i=1}^{n} (S^1_{a_i} \vee S^1_{b_i})$

mP^2 is represented by $a_1^2 \cdots a_m^2$.

Similarly $mP^2 \setminus \mathring{D}^2 \simeq \bigvee_{i=1}^{m} S^1_{a_i}$.

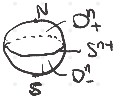
③ S^n $(n \geq 2)$ has trivial fundamental groups: $\pi_1(S^n) = 0$ for $n \geq 2$.

$S^n = D^n_+ \cup D^n_-$

$X_1 \cap X_2 = S^{n-1}$

$X_1 \cong \mathring{D}^n_+ \simeq \{N\}$

$X_2 \cong \mathring{D}^n_- \simeq \{S\}$



$D^n_+ = \{(x_1, \cdots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$

By the van Kampen theorem, $\pi_1(X) \cong \pi_1(X_1) * \pi_1(X_2)/N$
$= 0$

④ $S \cong S \# S^2$.

a surface

$X_1 = S \setminus \mathring{D}^2$

$X_2 = \mathring{D}^2$, $X_1 \cap X_2 \simeq S^1$

By the van Kampen theorem,

$\pi_1(S) \cong \pi_1(S \setminus \mathring{D}^2) * \pi_1(\mathring{D}^2)/N$.

$\cong \pi_1(S \setminus \mathring{D}^2) * e_2 / \langle i_{1\#}(\alpha)\, i_{2\#}(\alpha)^{-1} \rangle$

$S^1 \xrightarrow{f} S^1 \xrightarrow{i_1} X_1$, $S^1 \xrightarrow{f} S^1 \xrightarrow{i_2} X_2 \simeq *$

$\alpha = [f]$

$\cong \pi_1(S \setminus \mathring{D}^2) / \langle i_{1\#}(\alpha) \rangle$

(i) $S = nT^2$, $\pi_1(nT^2) \cong \pi_1(\bigvee_{i=1}^{n} (S^1_{a_i} \vee S^1_{b_i})) / \langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$

$i_\#(\alpha) \simeq [a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}]$

$= \langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \mid \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots, \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} \rangle$

(ii) $S = mP^2$,

$$\pi_1(mP^2) \cong \pi_1\left(\vee_{i=1}^{m} S^1_{a_i}\right) / \langle a_1^2 \cdots a_m^2 \rangle$$

$$\cong \langle \alpha_1, \cdots, \alpha_m \mid \alpha_1^2 \cdots \alpha_m^2 \rangle .$$

Abelianization of groups

$$G = \langle A \mid R \rangle$$

$$G^{ab} := \langle A \mid R \amalg \{ aba^{-1}b^{-1} \mid \forall a, b \in A \} \rangle$$

$$= G / [G,G], \quad [G,G] = \langle \{ aba^{-1}b^{-1} \mid a, b \in G \} \rangle .$$

Example. ① If G is commutative, $G^{ab} \cong G$. $[G,G] = 0$
$gh = hg$

② $F_n = \langle x_1, x_2, \cdots, x_n \rangle$

$$F_n^{ab} = \langle x_1, x_2, \cdots, x_n \mid x_i x_j = x_j x_i \rangle = \langle x_1, x_2, \cdots, x_n \mid \underbrace{x_i x_j x_i^{-1} x_j^{-1}}_{e}, \forall i, j \rangle$$

$$= \{ k_1 x_1 + k_2 x_2 + \cdots + k_n x_n \mid k_i \in \mathbb{Z} \}$$

$$\cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \ (n \text{ copies}) \cong \mathbb{Z}^n$$

More general, $G = C_1 * C_2 * \cdots * C_n$, C_i are cyclic groups,

$$G^{ab} \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n .$$

Basic property: If $G_1 \cong G_2$, then $G_1^{ab} \cong G_2^{ab}$.

$$(G_1^{ab} \not\cong G_2^{ab} \Rightarrow G_1 \not\cong G_2).$$

Application. $\pi_1(nT^2) \cong \langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \mid \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1} \rangle$

$$\pi_1(nT^2)^{ab} \cong \langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \mid \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_n \beta_n \alpha_n^{-1} \beta_n^{-1}, \boxed{\begin{array}{c} \alpha_i \beta_j \alpha_i^{-1} \beta_j^{-1} \\ \forall i, j \leq n \end{array}} \rangle$$

$$\cong \langle \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n \mid \alpha_i \beta_j \alpha_i^{-1} \beta_j^{-1}, \forall i, j \leq n \rangle$$

$$\Updownarrow$$

$$\alpha_i \beta_j = \beta_j \alpha_i$$

$$\cong \bigoplus_{i=1}^{n} \mathbb{Z}\langle \alpha_i \rangle \oplus \mathbb{Z}\langle \beta_i \rangle \cong \mathbb{Z}^{2n} .$$

Similarly, $\pi_1(mP^2)^{ab} \cong \langle \alpha_1,\dots,\alpha_m \mid \alpha_1^2\cdots\alpha_m^2,\ \alpha_i\alpha_j=\alpha_j\alpha_i\rangle$

$$\cong \langle \alpha_1,\dots,\alpha_m \mid 2(\alpha_1+\dots+\alpha_m)\rangle$$

$$\cong \langle \underbrace{\alpha_1+\dots+\alpha_m}_{\alpha_1'}, \alpha_2,\dots,\alpha_m \mid 2\underbrace{(\alpha_1+\dots+\alpha_m)}_{\alpha_1'},\ \alpha_i\alpha_j=\alpha_j\alpha_i\rangle$$

$$\cong \langle \alpha_1' \mid 2\alpha_1'\rangle \oplus \langle\alpha_2\rangle\oplus\cdots\oplus\langle\alpha_m\rangle$$

$$\cong \mathbb{Z}/2 \oplus \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} \cong \underline{\mathbb{Z}/2\oplus\mathbb{Z}^{m-1}}.$$

Clearly.
$$\left.\begin{array}{l} \pi_1(nT^2)^{ab}\not\cong\pi_1(n'T^2)^{ab} \text{ if } n'\neq n. \\ \pi_1(mP^2)^{ab}\not\cong\pi_1(m'P^2)^{ab} \text{ if } m'\neq m \\ \pi_1(nT^2)^{ab}\not\cong\pi_1(mP^2)^{ab},\ \forall m,n \end{array}\right\} \Rightarrow \begin{array}{l} \pi_1(nT^2)\not\cong\pi_1(n'T^2) \text{ if } n'\neq n \\ \pi_1(mP^2)\not\cong\pi_1(m'P^2) \text{ if } m'\neq m \\ \pi_1(nT^2)\not\cong\pi_1(mP^2),\ \forall m,n. \end{array}$$

$$\Rightarrow \left\{\begin{array}{ll} nT^2\not\cong n'T^2, & n\neq n' \\ mP^2\not\cong m'P^2, & m\neq m' \\ mP^2\not\cong nT^2, & \forall m,n. \end{array}\right. \ :$$

- Any two surfaces of different forms in $\{nT^2, mP^2\}$ are not homeomorphic. $\square$

Theorem (Brouwer's fixed point theorem)

Every map $f: D^n\to D^n$ has a fixed point: $\exists\, x_0\in D^n$, s.t. $f(x_0)=x_0$

proof of the case $n=2$.

Suppose that $\forall x\in D^2$, $f(x)\neq x$. $\quad x-f(x)\neq 0,\ \forall x\in D^2$.

Define $g(x)=\dfrac{x-f(x)}{\|x-f(x)\|}: D^2 \longrightarrow S^1$.

$h: S^1 \xrightarrow{i} D^2 \xrightarrow{g} S^1$. $\rightarrow$ $h\simeq id_{S^1}$!

Check that $h(x)\neq -x$, $\forall x\in\partial D^2$. (Exercise).

$\forall t\in[0,1]$, $H_t(x)=(1-t)h(x)+tx\neq 0$.

$G_t(x)=\dfrac{H_t(x)}{|H_t(x)|}: S^1\longrightarrow S^1$, give a homotopy $h\simeq id_{S^1}$.

$G_0(x)=H_0(x)=h(x)$, $G_1(x)=\dfrac{H_1(x)}{|H_1(x)|}=\dfrac{x}{\|x\|}=x$

Since $h = g \circ i \simeq \mathrm{id}_{S^1}$, $h_\# = g_\# \circ i_\#$ is an isomorphism, contradiction!

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_\#} & \pi_1(D^2) & \xrightarrow{g_\#} & \pi_1(S^1) \\ \| & & \| & & \| \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

Thus f has a fixed point. $\square$

Realization of groups:

For every group G, there is a space X_G st $\pi_1(X_G) \cong G$.

proof. Every group G is a quotient group of some free group.

$$G = \langle A \mid R \rangle = \langle x_\alpha \mid r_\beta \rangle \longleftarrow \langle x_\alpha \rangle = \pi_1(\vee_\alpha S^1_\alpha)$$

$$X_G = \underline{(\vee_\alpha S^1)} \cup_{\varphi_\beta} D^2_\beta \quad \varphi_\beta : \partial D^2_\beta = S^1_\beta \longrightarrow \vee_\alpha S^1_\alpha, \ [\varphi_\beta] = r_\beta.$$

Then $\pi_1(X_G) \cong \langle x_\alpha \mid r_\beta \rangle$. $\square$

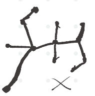
Contractible Spaces

A space X is <u>contractible</u> if $X \simeq \{*\}$.

Exercise. X is contractible $\Leftrightarrow X \simeq \{*\} \Leftrightarrow \mathrm{id}_X \simeq e_*$.

Examples. ① $\mathbb{R}^n \simeq \{*\}$, any convex subset of $\mathbb{R}^n$ is contractible.

② Every tree is contractible.



$\simeq *$

<u>Fact.</u> Let A be a contractible (closed) subset of X, then

$$\underline{X \simeq X/A.}$$

- Every graph G is homotopic to a wedge sum of S^1.

$G \simeq \vee_{i=1}^m S^1_i$, $m = \#\{\text{loops in } G\}$.

③ $S^\infty = \bigcup_{n=1}^\infty S^n$ is contractible.

proof. $S^\infty \subseteq \mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$. Want to show $id_{S^\infty} \simeq e_{x_0}$

$x = (x_1, x_2, \cdots)$, $|x| = 1$.

Let $\sigma_0 : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$, $\sigma_0(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$

Define $f_t(x) = (1-t)x + t\sigma_0(x) : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$.

$$\begin{array}{ccc} & \mathbb{R}^\infty & \longrightarrow & \mathbb{R}^\infty \\ & \cup\!\!\mid & & \cup\!\!\mid \\ & S^\infty & \longrightarrow & S^\infty \end{array}$$

Then $id_{\mathbb{R}^\infty} \overset{f_t}{\simeq} \sigma_0$

Define $g_t(x) = f_t(x) / |f_t(x)| : S^\infty \longrightarrow S^\infty$

$g_0 = id_{S^\infty}$, $g_1 = \sigma_0|_{S^\infty}$. $id_{S^\infty} \simeq \sigma_0|_{S^\infty}$

$\sigma_0|_{S^\infty}$, $S^\infty \longrightarrow S^\infty \hookrightarrow \mathbb{R}^\infty$

$e_{(1,0,0,\cdots)} : S^\infty \longrightarrow \{(1,0,0,\cdots) \in S^\infty\} \hookrightarrow \mathbb{R}^\infty$

$$h_t(x) = (1-t)\,\sigma_0|_{S^\infty} + t\, e_{(1,0,0,\cdots)}$$
$$= \big((1-t), tx_1, tx_2 \cdots\big)$$

$\therefore h_0 \simeq h_1$, $h_0 = \sigma_0|_{S^\infty} \simeq e_{(1,0,0,\cdots)}$

Thus $id_{S^\infty} \simeq e_{(1,0,0,\cdots)}$, S^∞ is contractible.