Lecture 09. Singular Homology Groups II.

Review of singular homology groups.

- Singular n-simplex $\sigma: \Delta^n = [v_0, \cdots, v_n] \longrightarrow X$

  $S_n(X) = \mathbb{Z}\langle \sigma: \Delta^n \to X \rangle$, the free abelian group generated by singular n-simplexes.

- $\partial\partial = 0$. $\cdots \to S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \cdots \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$

  $H_n(X) = \ker\partial_n / \mathrm{Im}\,\partial_{n+1}$. quotient group / module over $\mathbb{Z}$.

History: Poincaré $\longrightarrow$ Nöether (诺特)

- Induced homomorphism: $f: X \to Y$ induces $S_n(X) \xrightarrow{f_\#} S_n(Y)$ s.t. $\partial f_\# = f_\# \partial$.

  and hence induces $f_*: H_n(X) \longrightarrow H_n(Y)$. (recall the universal property of quotient group)

  $(gf)_* = g_* \cdot f_*$, $(id_X)_* = id_{H_n(X)}$, $\forall n$.

- Homotopy invariance: $f \overset{F}{\simeq} g: X \to Y$, then $f_* = g_*: H_n(X) \longrightarrow H_n(Y)$, $\forall n \geq 0$.

Correction. The fact that "$f \simeq g: X \to Y$ implies that $f_* = g_*: H_i(X) \to H_i(Y), \forall i \geq 0$"

$\Rightarrow f_\# - g_\# = \partial P + P\partial$, $P: C_i(X) \to C_{i+1}(Y)$

is less trivial. For the detailed proof, see Hatcher, AT, page 112.

Sketch proof of Homotopy invariance.

$f \overset{F}{\simeq} g: X \to Y$. $g_\#, f_\#: S_n(X) \longrightarrow S_n(Y)$

$F: X \times I \longrightarrow Y$ $\qquad \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$

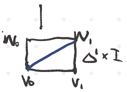
$\dim = n+1$

$\sigma: \Delta^n \longrightarrow X$

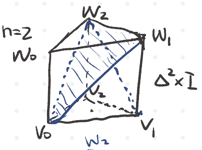
Δ^n

$n=0$

$n=1$



$n=2$



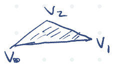
For $\Delta^n = [v_0, \cdots, v_n]$,

Set Δ^n_- as the bottom of $\Delta^n \times I$.

define $\Delta^n_+ = [w_0, \cdots, w_n]$ to be the top n-simplex of Δ^n s.t. the projection $\Delta^n_+ \longrightarrow \Delta^n_-$ satisfying

$w_i \longmapsto v_i$, $i = 0, \cdots, n$.

φ_2:



φ_{-1}:

For each i, define $\varphi_i: \Delta^n = [v_0, \cdots, v_n] \to I$

$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\}$

$\varphi_i(t_0, t_1, \cdots, t_n) = t_{i+1} + \cdots + t_n$.

$n=2$, $\Delta^n_- = [v_0, v_1, v_2]$

$\varphi_{-1}: \Delta^n_- \to 1$

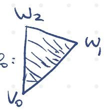
$\varphi_0(t_0, t_1, t_2) = t_1 + t_2$

$(0,1,0) \longmapsto 1$

$(1,0,0) \longmapsto 0$

$(0,0,1) \longmapsto 1$

φ_0:



$\varphi_{-1}(t_0, t_1, \cdots, t_n) = t_0 + t_1 + \cdots + t_n = 1$.

$\varphi_0(t_0, t_1, \cdots, t_n) = t_1 + \cdots + t_n$

$\varphi_1(t_0, t_1, \cdots, t_n) = t_2 + \cdots + t_n$

$\vdots$

$\varphi_n(t_0, t_1, \cdots, t_n) = 0$

$\varphi_1(v_0 = (1,0,0)) = 0$

$\varphi_1(v_1 = (0,1,0)) = 0$, $\varphi_2(v_2) = 1$

φ_1

In general, the followings hold:

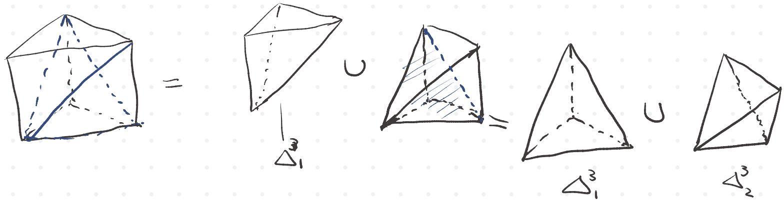
(i) $1 = \varphi_{-1} \geq \varphi_0 \geq \cdots \geq \varphi_n = 0$.

(ii) the image of φ_i is the n-simplex $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$

and $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ lies below the n-simplex $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$

(iii) the region between $[v_0, \cdots, v_i, w_{i+1}, \cdots, w_n]$ and $[v_0, \cdots, v_{i-1}, w_i, \cdots, w_n]$

forms an $(n+1)$-simplex $[v_0, \cdots, v_{i-1}, w_{i-1}, w_i, \cdots, w_n]$



In general, $\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \cdots, v_i, w_i, w_{i+1}, \cdots, w_n]$

any two $(n+1)$-simplexes intersect on one n-simplex

Define $P: S_n(X) \longrightarrow S_{n+1}(Y)$

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \, \underline{F \circ (\sigma \times id)\big|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]}} \in S_{n+1}(Y).$$

$$\Delta^{n+1} \longrightarrow \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y.$$

Check that $\underline{g_\# - f_\# = \partial P + P\partial}$. (Exercise)

Thus $g_* - f_* = 0$; i.e. $g_* = f_* : H_n(X) \longrightarrow H_n(Y)$, $\forall n \geq 0$.

$\square$

Cor. If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$, $\forall n$.

eg. $X \simeq \{*\}$ then $H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$

$D^n, \mathbb{R}^m, S^\infty$

Exact Sequences. 正合序列

A chain of (abelian) groups and homomorphism

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \xrightarrow{\phi_{n-1}} \cdots$$

is exact if $\mathrm{Im}\,\phi_{n+1} = \mathrm{Ker}\,\phi_n$, $\forall n$. ; or equivalent it is exact at all A_n.

exact at A_n if $\mathrm{Im}\,\phi_{n+1} = \mathrm{Ker}\,\phi_n$.

Examples. ① $A \xrightarrow{0} B \xrightarrow{b} C$ is exact $\Leftrightarrow$ b is mono.

② $B \xrightarrow{b} C \xrightarrow{0} D$ is exact $\Leftrightarrow$ b is epi.

③ $A \xrightarrow{0} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{0} E$ is exact $\Leftrightarrow \begin{cases} b \text{ is mono, } c \text{ is epi.} \\ C/\mathrm{Im}\,b \cong D. \end{cases}$

$\Updownarrow$

$0 \longrightarrow B \xrightarrow{b} C \xrightarrow{c} D \longrightarrow 0$ is exact. (short exact)

• note: Long exact sequenses induces short exact seq:

$$\begin{tikzcd}[column sep=small]
 & & & & & \mathrm{Im}\,\alpha_2 = \ker\alpha_3 \arrow[dr] & & & & \\
\cdots \arrow[r] & A_0 \arrow[r] & A_1 \arrow[rr, "\alpha_1"] \arrow[dr] & & A_2 \arrow[rr, "\alpha_2"] \arrow[ur] & & A_3 \arrow[rr, "\alpha_3"] \arrow[dr] & & A_4 \arrow[r] & \cdots \\
 & & & \mathrm{Im}\,\alpha_1 = \mathrm{Ker}\,\alpha_2 \arrow[ur] & & & & \mathrm{Im}\,\alpha_3 \arrow[ur] & &
\end{tikzcd}$$

exact seq.

$$0 \to \mathrm{Ker}\,\alpha_2 = \mathrm{Im}\,\alpha_1 \xrightarrow{\subseteq} A_2 \xrightarrow{\alpha_2} \mathrm{Im}\,\alpha_2 \longrightarrow 0$$

Axioms of (oridinary) homology theory / Eilenberg–Steenrod Axioms. 1945.

"Axiomatic approach to homology theory".

A_1: induced homomorphism. $f: X \to Y$ induces $f_*: H_n(X) \to H_n(Y)$.

A_2: $g_* f_* = (g \circ f)_*: H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z)$

A_3: $(\mathrm{id}_X)_* = \mathrm{id}: H_n(X) \longrightarrow H_n(X)$.

A_4: If $f \simeq g: X \to Y$, then $f_* = g_*: H_n(X) \longrightarrow H_n(Y)$.

A_5. Long exact sequences. Let ($A \subseteq X$ be a closed subset which is a deformation retraction of nbhd $V \subseteq X$: $V \underset{i}{\overset{r}{\rightleftarrows}} A$) — (X,A)—good pair. Then there is an exact seq:

reduced version $\longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$

unreduced version $\to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \to \cdots \to H_0(X,A) \to 0$

A_6: Excision: Given subspaces $Z \subseteq A \subseteq X$, s.t. $\bar{Z} \subseteq \operatorname{int}(A)$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces an isomorphism $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X, A)$, $\forall n \geq 0$.

another version: Set $B = X - Z$, $Z = X - B$, then $A - Z = A \cap B$
and $\bar{Z} \subseteq \operatorname{int}(A) \Leftrightarrow X = \operatorname{int}(A) \cup \operatorname{int}(B)$ (check).

for subspace $A, B \subseteq X$ s.t. $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, then the inclusion $(B, A \cap B) \longrightarrow (X, A)$ induces an isomorphism $H_n(B, A\cap B) \xrightarrow{\cong} H_n(X, A)$, $\forall n \geq 0$.

A_7: dimension axiom $H_n(\{p\}) = 0$ for $n > 0$.

- Moreover, any two homology theory h, H satisfy the above 7 axioms,

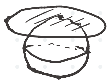
Uniqueness then $h = H$: $h_n(X) \cong H_n(X)$, $\forall X \in \mathrm{Top}_*$

- If A_7 axiom fails, then in general, $h_n(X) \not\cong H_n(X)$.

Theorem: $\longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0$

Recall $\tilde{H}_k(D^n) = 0$, $\forall k \geq 0$.

Example: $S^n \cong D^n / \partial D^n = S^{n-1}$. $n=2$



$(X, A) = (D^n, S^{n-1})$:

$$\tilde{H}_i(S^{n-1}) \longrightarrow \underset{\substack{\shortparallel \\ 0}}{\tilde{H}_i(D^n)} \xrightarrow{0} \underset{\substack{\cong \\ \tilde{H}_i(S^n)}}{\tilde{H}_i(D^n/S^{n-1})} \xrightarrow[\cong]{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{0} \underset{\substack{\shortparallel \\ 0}}{\tilde{H}_{i-1}(D^n)} \longrightarrow \cdots$$

$\partial D^1 = S^0$

Thus $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1}) \cong \cdots \cong \tilde{H}_{i-n}(S^0) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$

Relative homology groups $H_n(X,A)$

Given a pair of spaces (X,A), $A \subseteq X$ is a subspace.

$$\begin{array}{ccccccccc} 0 \to & S_n(A) & \xrightarrow{i_\#} & S_n(X) & \longrightarrow & S_n(X)/S_n(A) =: S_n(X,A) & \to 0 & & \text{exact} \\ & \downarrow \partial_A & & \downarrow \partial_X & & \downarrow \tilde{\partial} & & & \\ 0 \to & S_{n-1}(A) & \xrightarrow{i_\#} & S_{n-1}(X) & \longrightarrow & S_{n-1}(X)/S_{n-1}(A) = S_{n-1}(X,A) & \to 0 & & \end{array}$$

$$\partial_X \partial_X = 0, \; \partial_A \partial_A = 0 \Rightarrow \tilde{\partial}\tilde{\partial} = 0.$$

The quotient group $\operatorname{Ker}\tilde{\partial} / \operatorname{Im}\tilde{\partial} =: H_n(X,A)$ is called the n-th relative homology group of the pair (X,A).

- If $A = \phi$, then $H_n(X,\phi) = H_n(X)$.
- Exercise: describe elements of $\operatorname{Ker}\tilde{\partial}$ and $H_n(X,A)$.
- $H_n(X,A) = 0, \forall n \Leftrightarrow H_n(A) \cong H_n(X), \forall n$.

Prop. For "good pair" (X,A), there holds an isomorphism

$$H_n(X,A) \cong \tilde{H}_n(X/A), \; \forall n \geq 0.$$

Cor. For good pair (X,A), there is an exact sequence.

$$\to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots \to H_0(X,A) \to 0$$

Example (Exercise) ① Compute $H_k(D^n, S^{n-1})$.

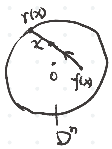
② proof of Brouwer's fixed point theorem.

Every map $f: D^n \to D^n$ $(n \geq 1)$ has a fixed point ($f(x) = x$, $x \in D^n$).

$n=2$, proved by fundamental groups $\pi_1(S^1) \cong \mathbb{Z}$.

Recall that we constructed a map $r: D^n \to \partial D^n = S^{n-1}$ s.t. the composition

$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ is the identity: $ri = id_{S^{n-1}}$.



$$(ri)_* = r_* i_* = id: \quad \begin{array}{ccccc} \tilde{H}_{n-1}(S^{n-1}) & \longrightarrow & \tilde{H}_{n-1}(D^n) & \longrightarrow & \tilde{H}_{n-1}(S^{n-1}) \\ \| & & \| & & \| \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \end{array}$$

Contraction: $\mathbb{Z} \to 0 \to \mathbb{Z}$ cannot be the identity. $\square$

Long Exact Sequence II

Mayer-Vietoris Sequence: (MV sequence)

Let $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, $A, B \subseteq X$. There is an exact seq:

$$\xrightarrow{\partial} \widetilde{H}_n(A\cap B) \xrightarrow{\Phi} \underset{(\alpha,\beta)}{\widetilde{H}_n(A) \oplus \widetilde{H}_n(B)} \xrightarrow{\Psi} \widetilde{H}_n(X) \xrightarrow{\partial} \widetilde{H}_{n-1}(A\cap B) \to \cdots$$

$$\Phi(\alpha) = (\alpha, -\alpha)$$
$$\Psi(\alpha,\beta) = \alpha+\beta.$$

Example: ① $S^n = D^n_+ \underset{S^{n-1}}{\cup} D^n_-$ $\Rightarrow \widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1}) \cong \cdots \cong \widetilde{H}_{i-n}(S^0) \cong \begin{cases} \mathbb{Z} & i=n \\ 0 & i\neq n \end{cases}$

② $\widetilde{H}_n(X_1 \vee X_2) \cong \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \Rightarrow \widetilde{H}_n(\bigvee_{i=1}^m X_i) \cong \bigoplus_{i=1}^m \widetilde{H}_i(X_i)$

$X_1 \vee X_2 = \dfrac{(X_1, x_1) \amalg (X_2, x_2)}{x_1 \sim x_2}$ eg. $\widetilde{H}_1(S^1 \vee S^1) \cong \mathbb{Z}\oplus\mathbb{Z}$.

$S^1 \vee S^1$

③ T^2

$S^1 \vee S^1$

$A \cap B = S^1$,
$A \cong \mathring{D}^2$
$B = T^2 \setminus D^2 \simeq S^1 \vee S^1$

$$0 \to \underset{\overset{\|}{0}}{\widetilde{H}_2(S^1)} \to \underset{\overset{\|}{0}}{\widetilde{H}_2(D^2)} \oplus \underset{\overset{\|}{\widetilde{H}_2(S^1)\oplus\widetilde{H}_2(S^1)} \overset{\|}{0}}{\widetilde{H}_2(S^1\vee S^1)} \to \underset{\overset{\|}{?}}{\widetilde{H}_2(T^2)} \xrightarrow{\partial} \underset{\overset{\|}{\mathbb{Z}}}{\widetilde{H}_1(S^1)} \to \underset{\overset{\|}{\mathbb{Z}\oplus\mathbb{Z}}}{\widetilde{H}_1(D^2) \oplus \widetilde{H}_1(S^1\vee S^1)} \to \underset{\overset{\|}{?}}{\widetilde{H}_1(T^2)} \to \underset{}{0}$$

$$\therefore\ 0 \to \underset{\mathbb{Z}}{\widetilde{H}_2(T_2)} \xrightarrow{a} \mathbb{Z} \xrightarrow{b} \mathbb{Z}\oplus\mathbb{Z} \to \widetilde{H}_1(T^2) \to 0$$

$\operatorname{Im} a \cong \mathbb{Z}$, $\operatorname{Im} b$

$0 \neq \widetilde{H}_2(T^2) \subseteq \mathbb{Z} \Rightarrow \widetilde{H}_2(T^2) \cong \mathbb{Z}$.

$0 \to \mathbb{Z} \cong \operatorname{Im} a \to \mathbb{Z} \to \operatorname{Im} b \to 0$ exact $\Rightarrow \operatorname{Im} b$ is finite ($\operatorname{Im} b \cong \mathbb{Z}/n\mathbb{Z}$ or $\operatorname{Im} b = 0$) $\xrightarrow{\operatorname{Hom}(\mathbb{Z}/n,\, \mathbb{Z}\oplus\mathbb{Z}) = 0}$ $0 \to \operatorname{Im} b \xrightarrow{0} \mathbb{Z}\oplus\mathbb{Z} \to \widetilde{H}_1(T^2) \to 0$ exact $\Rightarrow \widetilde{H}_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$.

④ Exercise. $\widetilde{H}_n(K)$, K is the klein bottle.

$\cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & n=1 \\ 0 & n\neq 1 \end{cases}$

Exam Time: 8.5. Next Saturday