Lecture 10 Singular Homology Groups III

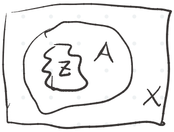
Recall $H_n(X,A)$. ① $H_n(X,\phi) = H_n(X)$

② $H_n(X,x_0) = \tilde{H}_n(X)$

③ Long exact seq: $\rightarrow H_n(X,A)=0, \forall n \Leftrightarrow H_n(A) \xrightarrow[\cong]{i_*} H_n(X)$

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_{n*}} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \rightarrow \cdots \quad \rightarrow H_0(X,A).$$

Excision Theorem Given subspaces $Z \subseteq A \subseteq X$, s.t. $\bar{Z} \subseteq \text{int}(A)$, then the



inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces an isomorphism $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$, $\forall n \geq 0$.

Application

Theorem (Invariance of domains)

Let $\phi \neq U \subseteq \mathbb{R}^m$, $\phi \neq V \subseteq \mathbb{R}^n$ be open subsets.

If U is homeomorphic to V, then $m=n$.

In particular, $\mathbb{R}^m \cong \mathbb{R}^n \Rightarrow m=n$. ($m \neq n$, $\mathbb{R}^m \not\cong \mathbb{R}^n$, $\mathbb{R}^m \simeq \mathbb{R}^n$)

proof. Let $x \in U$. $(\underbrace{U}_{X-Z}, U-\{x\}) \hookrightarrow (\mathbb{R}^m, \mathbb{R}^m-\{x\})$

$Z = \mathbb{R}^m - U$. $U = \mathbb{R}^m - Z$.

induces an isomorphism

$$\underline{H_k(U, U-\{x\})} \cong H_k(\mathbb{R}^m, \mathbb{R}^m-\{x\})$$

$\tilde{H}_k(\mathbb{R}^m) = 0, \forall k$. $\underset{\overset{\|}{0}}{\tilde{H}_k(\mathbb{R}^m)} \rightarrow H_k(\mathbb{R}^m, \mathbb{R}^m-\{x\}) \xrightarrow{\partial} \tilde{H}_{k-1}(\mathbb{R}^m-\{x\}) \rightarrow \underset{\overset{\|}{0}}{\tilde{H}_{k-1}(\mathbb{R}^m)}$

$$\therefore H_k(\mathbb{R}^m, \mathbb{R}^m-\{x\}) \cong \tilde{H}_{k-1}(\mathbb{R}^m-\{x\}) \cong \tilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m. \end{cases}$$

Let $h: U \rightarrow V$ be a homeomorphism, $h: (U, U-\{x\}) \xrightarrow{\cong} (V, V-\{h(x)\})$

$$h_*: \underset{\begin{cases} \mathbb{Z} & k=m \\ 0 & k\neq m \end{cases}}{\underset{\cong}{H_k(U, U-\{x\})}} \xrightarrow{\cong} \underset{\begin{cases} \mathbb{Z} & k=n \\ 0 & k\neq n \end{cases}}{\underset{\cong}{H_k(V, V-\{h(x)\})}}$$

Then h_* is an isomorphism implies that $m=n$. $\square$

- **Lemma.** Let $U \subseteq \mathbb{R}^m$ be a nonempty open subset. $x \in U$.

  Then $H_k(U, U-x) \cong \begin{cases} \mathbb{Z} & k=m \\ 0 & k \neq m. \end{cases}$

Degree of selfmaps of $\underline{S^n}$.

映射度

Let $f: S^n \to S^n$, $n \geq 0$, $H_n(f): \tilde{H}_n(S^n) \longrightarrow \tilde{H}_n(S^n)$, where $\tilde{H}_n(S^n) \cong \mathbb{Z} \ni 1$ and $\tilde{H}_n(S^n) \cong \mathbb{Z}$.

$H_n(f)(1) =: \deg(f)$, is called the degree of f.

$H_n(f)(\alpha) = \deg(f) \cdot \alpha$, $\alpha \in \mathbb{Z}$ is a generator.

Proposition. The followings hold:

(a) $f = id: S^n \to S^n$, $\deg(id) = 1$

(b) $\deg(g \circ f) = (\deg g) \cdot (\deg f)$: $(gf)_* = g_* f_*$

(c) $\deg g = \deg f$ if $f \simeq g$: $f_* = g_*$ $\Rightarrow$ $\deg f$ is a homotopy invariant.

$$\deg: [S^n, S^n] \longrightarrow \mathbb{Z}, \quad [f] \mapsto \deg f.$$

(4) $\deg f = -1$ if f is a reflection of S^n.

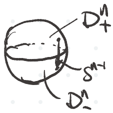
$$S^n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \textstyle\sum_{i=1}^{n+1} x_i^2 = 1\}$$

A reflection $r_i: S^n \to S^n$ is defined by

$$r_i(x_1, \cdots, x_i, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, -x_i, x_{i+1}, \cdots, x_{n+1})$$

$r_{n+1}: S^n \to S^n$.

$\deg r_{n+1} = -1$ $\quad H_n(S^n) \longrightarrow H_n(S^n)$, $\quad H_n(S^n) \cong \mathbb{Z}\langle D_+^n - D_-^n \rangle$ (fact)

S^n 

$$r_*(D_+^n - D_-^n) = D_-^n - D_+^n = -(D_+^n - D_-^n)$$

(5) $-1: S^n \to S^n$, $(x_1, x_2, \cdots, x_{n+1}) \mapsto (-x_1, -x_2, \cdots, -x_{n+1})$

$-1 = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$ $\quad \therefore$ $\deg(-1) = (-1)^{n+1}$.

(6) If f is not surjective, then $\deg f = 0$. (exercise)

Local degree. $f: S^n \to S^n$, $n>0$

$\exists\, y \in S^n$, st $f^{-1}(y) = \{x_1, x_2, \cdots, x_m\}$

S^n is Hausdorff, let $U_1, U_2, \cdots, U_m$ be nbhds of $x_1, x_2, \cdots, x_m$, respectively

St. $U_i \cap U_j = \Phi$, $\forall i \neq j$.

Let V be an nbhd of y st. $f(U_i) \subset V$, $i=1,2,\cdots,m$.

Consider the following commutative diagram:

$1 \longmapsto \deg f|x_i$ — local degree

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dr, phantom, "① \ \text{excision}"] & H_n(V, V-y) \cong \mathbb{Z} \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \cong \mathbb{Z}^m \arrow[l, "p_i"'] \arrow[r, "f'_*"] \arrow[dr, phantom, "②"] & H_n(S^n, S^n - y) \cong \mathbb{Z} \\
 & H_n(S^n) \arrow[u, "j_*"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong", "j_*"']
\end{tikzcd}$$

$j_*(1) = (1, 1, \cdots, 1)$

$1 = j_* p_i(1)$

$S^n - y \cong \mathbb{R}^n$

$1 \longmapsto \deg f$

Lemma: $H_n(\amalg_{i=1}^m U_i, \amalg_{i=1}^m (U_i - x_i)) \xrightarrow{\cong} H_n(S^n, S^n - f^{-1}(y))$. (Excision theorem)

$$\begin{tikzcd}
H_n(\amalg_{i=1}^m U_i, \amalg_{i=1}^m (U_i - x_i)) \arrow[r, "\cong"] \arrow[d, phantom, "\cong"] & H_n(S^n, S^n - f^{-1}(y)) \arrow[d, "p_i'"] \arrow[dr, "p_i"] & \\
\bigoplus_{i=1}^m H_n(U_i, U_i - x_i) \arrow[ur, "\oplus_{i=1}^m k_i", "\cong"'] \arrow[d, phantom, "\cong"] & H_n(U_i, U_i - x_i) \arrow[r, "\cong"] & H_n(S^n, S^n - x_i) \\
\bigoplus_{i=1}^m \mathbb{Z} & &
\end{tikzcd}$$

By commutativity, $j_*(1) = (1, 1, \cdots, 1)$, $k_i(1) = 1$, $i = 1, \cdots, m$.

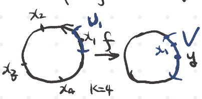
By ①, $f'_* k_i(1) = \deg f|x_i$

By ②, $\deg f = f'_* j_*(1) = f'_*(1, 1, \cdots, 1) = f'_*(k_1(1), k_2(1), \cdots, k_m(1)) = \sum_{i=1}^m \deg f|x_i$

$\boxed{\text{Prop. } \deg f = \sum_i \deg f|x_i}$

Example. $f: S^1 \to S^1$, $z \mapsto z^k$, $k \in \mathbb{Z}$, has degree k.

proof. It suffices to prove it when $k>0$.



$\deg f = \sum_{i=1}^k \deg f|x_i$.

$\forall y \in S^1$, $\exists\, x_1, \cdots x_k \in S^1$ st. $f(x_i) = y$, $i = 1, 2, \cdots, k$.

$U_i \xrightarrow{f} V$

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$f|U_i : U_i \to V$ is a homeomorphism preserving orientation.

$f|U_i \simeq id$. $\deg f|U_i = 1$

$\therefore \deg f = \sum_{i=1}^k \deg f|x_i = k$. □

Cone The cone CX on a topological space X is the quotient space

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$$CX = X\times I \big/ X\times\{1\}.$$

CX is contractible. $\tilde{H}_n(CX)=0,\ \forall n\geq 0.$

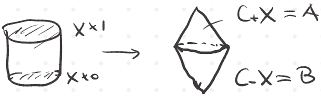
Suspension $\Sigma X = CX \cup_X CX$

双角锥

$$= \frac{X\times I}{X\times\{0\},\ X\times\{1\}}$$

Examples: $CS^n \cong D^{n+1}$ (take $n=1$)

$\Sigma S^n \cong S^{n+1}$ (——)



$A\cap B = X.$

Exercise / Lemma. There is an isomorphism $H_{n+1}(\Sigma X) \xrightarrow[\cong]{\partial} H_n(X)$.

(Mayer–Vietoris Sequence)

• Fact: Any $f: X\to Y$ induces maps $Cf: CX \to CY$

$\Sigma f: \Sigma X \longrightarrow \Sigma Y.$

prop. Let $f: S^n \to S^n$. Then $\deg(\Sigma f) = \deg f$.

proof.

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] & H_n(S^n) \\
H_n(\Sigma S^n) \arrow[u, "\cong"', "\partial"] \arrow[r, "(\Sigma f)_*"] & H_n(\Sigma S^n) \arrow[u, "\cong"', "\partial"]
\end{tikzcd}$$

commutes.

$\therefore \deg(\Sigma f) = \deg f.$ $\square$

$S^1 \xrightarrow{f_k} S^1,\ z\mapsto z^k.$

$\Sigma f: S^2 \to S^2,\ \deg(\Sigma f) = k$

$\Sigma^2 f = \Sigma(\Sigma f): S^3 \to S^3,\ \deg(\Sigma^2 f) = k.$

$\Rightarrow$ $\deg: [S^n, S^n] \longrightarrow \mathbb{Z}$ is surjective.

CW complexes / Cell complexes $\longrightarrow$ for cellular homology.
胞腔同调

Def. A set X is a CW complex or cell complex if there is a chain of subsets

$$\phi = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

s.t. (i) $X^0 = \{\text{discrete points of } X\}$ — 0-cells.

(ii) $X^n = X^{n-1} \sqcup_\alpha D^n_\alpha \big/ x \sim \varphi_\alpha(x),\ \forall x \in \partial D^n_\alpha = S^{n-1}_\alpha$

$= X^{n-1} \cup_{\varphi_\alpha} D^n_\alpha \qquad e^n_\alpha \cong \mathrm{int}(D^n)$ — n-cells of X.

$\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$ is called the attaching map of e^n_α.

φ_α can be extends to a map $\Phi_\alpha : (D^n_\alpha, \partial D^n_\alpha) \longrightarrow (X^n, X^{n-1})$

Φ_α is a homeomorphism when restricted to $\mathrm{int}(D^n_\alpha)$.

$e^n_\alpha = \Phi_\alpha(\mathrm{int}(D^n_\alpha))$.

(iii) $X = \bigcup_{i=0}^{n \leq \infty} X^n$. X^n is called the n-skeleton of X.

(iv) X has weak topology — W. $\qquad X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$

$V \subseteq X$ is open (or closed)

$\Leftrightarrow V \cap X^n \subseteq X^n$ is open (or closed).

$\Leftrightarrow V \cap e^n_\alpha \subseteq e^n_\alpha$ is open (or $V \cap \overline{e^n_\alpha} \subseteq \overline{e^n_\alpha}$ is closed).

"C" — the closure of each e^n_α meets only finitely many other cells.

Closure-finiteness

Examples. ① $D^n = \mathrm{int}(D^n) \sqcup \partial D^n = e^n \cup S^{n-1} = e^n \cup e^{n-1} \cup e^0$

$S^n = e^0 \cup e^n \qquad \rightarrow D^n/\partial D^n = \dfrac{e^0 \cup e^{n-1} \cup e^n}{e^0 \cup e^{n-1}} \cong e^0 \cup e^n \cong S^n$.

$\{\infty\} \cup \mathbb{R}^n$

② $nT^2 = T^2 \# \cdots \# T^2$ (n copies)

$= \left(\bigvee_{i=1}^n (S^1_{a_i} \vee S^1_{b_i}) \right) \cup_\varphi D^2$

$= e^0 \cup (e^1_{a_1} \cup e^1_{b_1}) \cup \cdots \cup (e^1_{a_n} \cup e^1_{b_n}) \cup e^2$.

$\varphi_{nT^2} : S^1 \longrightarrow \bigvee_{i=1}^n (S^1_{a_i} \vee S^1_{b_i}) = X^1$

$S^1 \longmapsto a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}$

$\varphi : S^1 \to a_1^2 a_2^2 \cdots a_m^2$.

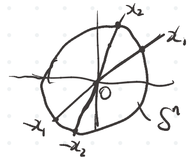
$mP^2 = (\bigvee_{i=1}^m S^1_{a_i}) \cup_\varphi D^2 = e^0 \cup (e^1_{a_1} \cup \cdots \cup e^1_{a_m}) \cup e^2$.

$X = \bigcup_{n,\alpha} e^n_\alpha$ is called a cell decomposition of X.

③ $\mathbb{R}P^n = \mathbb{R}^{n+1}\setminus\{0\} \Big/ {x\sim\lambda x,\ \lambda\in\mathbb{R}\setminus\{0\} \atop x\in\mathbb{R}^{n+1}}$

real projective space $\quad = S^n \big/ x\sim -x,\ \forall x\in S^n$

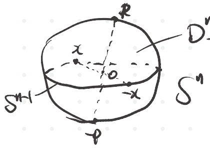
$P^2 = \mathbb{R}P^2.$



$= D_+^n \big/ x\sim -x,\ x\in\partial D^n = S^{n-1}$

$= e^n \cup \mathbb{R}P^{n-1} = \cdots = \underline{e^n\cup e^{n-1}\cup\cdots\cup e^1\cup e^0} = \mathbb{R}P^n$

in particular, $\mathbb{R}P^0 = e^0$, $\mathbb{R}P^1 = e^1\cup e^0 = S^1$.



$D_+^n = \{(x_1,\cdots,x_{n+1})\in S^n \mid x_{n+1}\geq 0\}.$

④ $\mathbb{C}P^n = \underset{\cong\ \mathbb{R}^{2n+2}}{\mathbb{C}^{n+1}}\setminus\{0\} \Big/ {z\sim\lambda z,\ z\in\mathbb{C}^{n+1}\setminus\{0\} \atop \lambda\in\mathbb{C}\setminus\{0\}}$

complex projective space

$= S^{2n+1} \big/ z\sim\lambda z,\ |\lambda|=1$

$= D_+^{2n} \big/ z\sim\lambda z,\ |\lambda|=1,\ z\in\partial D^{2n} = S^{2n-1}$

$= e^{2n}\cup\mathbb{C}P^{n-1} = \cdots = \underline{e^{2n}\cup e^{2n-2}\cup\cdots\cup e^2\cup e^0} = \mathbb{C}P^n$

$\Rightarrow \mathbb{C}P^1 = S^2.$

$S^{2n+1} \subseteq \mathbb{C}^{n+1}$

$\{(z_1,z_2,\cdots,z_{n+1})\in\mathbb{C}^{n+1} \mid |z_1|^2+\cdots+|z_{n+1}|^2=1\}.$

$A = \{(z_1,z_2,\cdots,z_{n+1})\in S^{n+1} \mid z_{n+1}\geq 0\}$

$= \{(z_1,z_2,\cdots,z_{n+1})\in S^{n+1} \mid z_{n+1} = \sqrt{1-|z_1|^2-\cdots-|z_n|^2}\}$

$= \{(w,\sqrt{1-|w|^2})\in\mathbb{C}^n\times\mathbb{C} \mid |w|\leq 1\} = D^{2n}$

$\partial A = \{(w,\sqrt{1-|w|^2})\in\mathbb{C}^n\times\mathbb{C} \mid |w|=1\}$

$= \{(w,0)\in\underset{\overset{\shortparallel}{\mathbb{R}^{2n}}}{\mathbb{C}^n}\times\mathbb{C} \mid |w|=1\} = S^{2n-1}$

Time: 8.5. Saturday.

开卷. 2小时.