Lecture 11. Cellular Homology Groups ·胞腔同调

CW complexes $X = \bigcup_{i=0}^{a\le\infty} X^n$, X^n — n-skeleton of X

$X = nT^2,\ mP^2,\ \mathbb{R}P^n,\ \mathbb{C}P^n$

$X^{n-1}, X^1, \cdots$

Lemma. Let X be a CW complex. Then

(i) $H_k(X^n, X^{n-1}) = \begin{cases} \mathbb{Z}^{\phi(n)}, & k=n, \quad \phi(n) = \#\{e^n_\alpha\} \\ 0, & k\neq n. \end{cases}$

$\left(\begin{array}{l} X^n = X^{n-1} \sqcup_\alpha e^n_\alpha \\ H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) \cong \tilde{H}_k(\vee_\alpha S^n_\alpha) \end{array} \right)$

(ii) $H_k(X) = 0$ for $k > \dim X$ if X is of finite dimension. $X^{\dim X} = X$

(iii) the inclusion map $i_n : X^n \hookrightarrow X$ induces an isomorphism

$$i_{n*} : H_k(X^n) \to H_k(X) \text{ for } k<n,$$

and an epimorphism $i_{n*} : H_n(X^n) \to H_n X$.

proof. Exercise.

Rmk: By (i) of the Lemma, we usually say $H_n(X^n, X^{n-1})$ is generated by its n-cells.

Construction

$$\begin{tikzcd}
 & 0 \arrow[dr] & & H_n(X) \arrow[r] & 0 & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}=j_n\circ\partial_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"] \arrow[rr, "d_n=j_{n-1}\circ\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] \arrow[dr, two heads] & & \\
 & & & 0 \arrow[ur] & & H_{n-1}(X) &
\end{tikzcd}$$

$H_n(X) \cong H_n(X^n)/\mathrm{Im}\,\partial_{n+1}$

Observation: $d_n \circ d_{n+1} = 0$; $\partial_n j_n = 0$

Define $C_n(X) = H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle e^n_\alpha \rangle$

$$\cdots \to C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X) \to \cdots \xrightarrow{d_1} C_0(X) \to 0$$

$H_n^{CW}(X) := \dfrac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}$ is called the n-th cellular homology group of the cell complex X.

Prop. Let X be a CW complex. Then there is an isomorphism

$$H_n^{CW}(X) \cong H_n(X).$$

proof. By the exactness of " $\nearrow^0$ " we have an isomorphism

$$H_n(X^n)/\mathrm{Im}\,\partial_{n+1} \xrightarrow{\cong} H_nX.$$

$$\cong \Big\downarrow \tilde{j}_n$$

$$\frac{\mathrm{Im}\, j_n = \mathrm{Ker}\,\partial_n = \mathrm{ker}\, d_n}{j_n(\mathrm{Im}\,\partial_{n+1}) = \mathrm{Im}\, d_{n+1}} = H_n^{CW}(X)$$

Since j_n is injective, we have

$$j_n(\mathrm{Im}\,\partial_{n+1}) = \mathrm{Im}(j_n \circ \partial_{n+1}) = \mathrm{Im}\, d_{n+1}$$

By the exactness of " $\searrow$ ", we have

$$\mathrm{Im}\, j_n = \ker \partial_n.$$

Since j_{n-1} is injective, we have

$$\ker d_n = \ker(j_{n-1} \circ \partial_n) = \ker \partial_n.$$

Thus we get an isomorphism $H_n^{CW}(X) \xrightarrow{\cong} H_n(X)$. $\square$

Applications. $H_n^{CW}(X) \cong H_n(X)$, $C_n(X) \cong \mathbb{Z}\langle e_\alpha^n \rangle \xrightarrow{d_n} C_{n-1}(X) \cong \mathbb{Z}\langle e_\beta^n \rangle$

① If $k > \dim X$, $H_k(X) = 0$; or more general, if X has no n-cells, then $H_n(X) = 0$.

② If X is a CW complex having no two cells in adjacent dimensions, then $H_n(X)$ is either 0 or is isomorphic to $C_n(X)$.

Eg. $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$,

$$H_k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & k = 0, 2, \cdots, 2n; \\ 0 & \text{otherwise.} \end{cases}$$

③ $H_0(X) \cong \mathbb{Z}$ if X is path-connected.

$$\overset{\cong}{H_0^{CW}(X)} = \frac{C_0(X) = \mathbb{Z}}{\mathrm{Im}\, d_1} \cong \mathbb{Z} \qquad C_1(X) \xrightarrow{d_1 = 0} \underset{\mathbb{Z}}{\overset{}{C_0(X)}} \to 0$$

Further assume that X has a unique 0-cell, then $H_0(X) \cong \mathbb{Z}$ implies that $d_1 = 0$.

④ The Euler-Poincaré characteristic of a CW complex:

$$\chi(X) = \sum_n (-1)^n \#\{e_\alpha^n\} = \sum_n (-1)^n \operatorname{rank} C_n(X).$$

For an abelian group A, $\operatorname{rank}(A) = r$ if $\mathbb{Z}^r \hookrightarrow A$ and $\mathbb{Z}^{r+1} \not\hookrightarrow A$.

$$\operatorname{rank}(\mathbb{Z}^n) = \operatorname{rank}(\mathbb{Z}^n \oplus \mathbb{Z}/m) = n.$$

Lemma. (i) If B is a subgroup or a quotient group of an abelian group A, then $\mathrm{rank}(B) \leq \mathrm{rank}(A)$

(ii) If there is a short exact seq. of abelian groups;

$$0 \longrightarrow A \to B \to C \to 0,$$

if $\mathrm{rank}(B) < \infty$, then $\mathrm{rank}(B) = \mathrm{rank}(A) + \mathrm{rank}(C)$.

proof. Exercise.

Let X be a finite CW complex.

prop. $\chi(X) = \sum_n (-1)^n \,\mathrm{rank}\, C_n(X) = \sum_n (-1)^n \,\underline{\mathrm{rank}\, H_n(X)}$.

$\Rightarrow$ The Euler-Poincaré characteristic is a homotopy invariant.

proof of $\chi = \sum_n (-1)^n \,\mathrm{rank}\, C_n = \sum_n (-1)^n \,\mathrm{rank}\, H_n$.

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

Let $\ker d_n = Z_n$, $\mathrm{Im}\, d_{n+1} = B_n$. then $H_n = Z_n/B_n \Leftrightarrow 0 \to B_n \to Z_n \to H_n \to 0$ is exact.

$0 \to \underset{\underset{Z_n}{\parallel}}{\ker d_n} \longrightarrow C_n \xrightarrow{d_n} \mathrm{Im}\, d_n = B_{n-1} \to 0$ is exact.

By the Lemma above, $\mathrm{rank}\, Z_n = \mathrm{rank}\, B_n + \mathrm{rank}\, H_n$

$\mathrm{rank}\, C_n = \mathrm{rank}\, Z_n + \mathrm{rank}\, B_{n-1}$

$\Rightarrow \mathrm{rank}\, C_n = \mathrm{rank}\, H_n + \mathrm{rank}\, B_n + \mathrm{rank}\, B_{n-1}$.

$$\sum_n (-1)^n \,\mathrm{rank}\, C_n = \sum_n (-1)^n \,\mathrm{rank}\, H_n. \qquad \square$$

prop. (Cellular Boundary Formula)

Let $d_n : C_n(X) \longrightarrow C_{n-1}(X)$ be the boundary homomorphism. Then

$$\boxed{d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta}$$

where $d_{\alpha\beta}$ is the degree of the composition

$$\underline{\Delta_{\alpha\beta} : \partial D^n_\alpha = S^{n-1}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} = \vee_\beta S^{n-1}_\beta \xrightarrow{q_\beta} S^{n-1}_\beta.}$$

recall e^n_α has the characteristic map $\Phi_\alpha : D^n_\alpha \longrightarrow X^n$, $\Phi_\alpha|_{\partial D^n_\alpha} = \varphi_\alpha$.

proof. Consider the following comm. diagram induced by Φ_α:

$$\begin{tikzcd}
{[D^n_\alpha]} \arrow[dd, bend right, mapsto] & H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial_n"] \arrow[d, "\Phi_{\alpha *}"] & \tilde{H}_{n-1}(\partial D^n_\alpha = S^{n-1}_\alpha) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
& H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[ddr, "d_n"] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[dd, "j_{n*}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"] \\
e^n_\alpha & C_n(X) & & \tilde{H}_{n-1}(\vee_\beta S^{n-1}_\beta) \\
& & H_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}(X) &
\end{tikzcd}$$

$\partial_n[D^n_\alpha] = [\partial D^n_\alpha]$, $\quad H_n(X^n, X^{n-1}) = C_n(X) \ni e^n_\alpha$, $\quad \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \cong \tilde{H}_{n-1}(\vee_\beta S^{n-1}_\beta)$

By the commutativity of the left square,

$$d_n(e^n_\alpha) = d_n \Phi_{\alpha *}[D^n_\alpha] = j_{n*}\varphi_{\alpha *}\partial_n[D^n_\alpha]$$

Since $q_{\beta *}$ is the projection, we get

$$\begin{aligned} \varphi_* \partial_n[D^n_\alpha] &= \Sigma_\beta \Delta_{\alpha\beta *}(\partial[D^n_\alpha]) \\ &= \Sigma_\beta d_{\alpha\beta} e^{n-1}_\beta. \end{aligned}$$

$\square$

Examples. ① $nT^2 = T^2 \# \cdots \# T^2$ (n copies)

$$X^1 = V_n(S^1 \vee S^1) = e^0 \cup \bigcup_{i=1}^n (e^1_{a_i} \cup e^1_{b_i})$$

$$X^2 = nT^2.$$

$H_0(nT^2) \cong \mathbb{Z}$, $H_k(nT^2) = 0$ for $k > 2$.

$$0 \to C_2(nT^2) \xrightarrow{d_2} C_1(nT^2) \xrightarrow{d_1 = 0} C_0(nT^2)$$

$$0 \to \mathbb{Z}e^2 \xrightarrow{d_2} \mathbb{Z}^{2n}\langle e^1_{a_1}, e^1_{b_1}, \cdots, e^1_{a_n}, e^1_{b_n} \rangle$$

$$\partial D^2 = S^1 \xrightarrow{\varphi} X^1 = \bigvee_{i=1}^n (S^1_{a_i} \vee S^1_{b_i}) \xrightarrow{q_{a_i}} S^1_{a_i}$$

$$\varphi = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}.$$

Since $\pi_1 S^1 \cong \mathbb{Z}$, $q_{a_i} \circ \varphi$ is nullhomotopic (零伦: homotopic to the constant map).

$$d_{a_i} = 0$$

$\therefore d_2 = 0. \Rightarrow H_2(nT^2) \cong \mathbb{Z}\langle e^2 \rangle$

$$H_1(nT^2) \cong \mathbb{Z}^{2n}.$$

② Compute $H_i(mP^2)$. Exercise.

③ $H_i(\mathbb{RP}^n)$.

$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n = e^0 \cup e^1 \cup \cdots \cup e^{n-1} \cup e^n$.

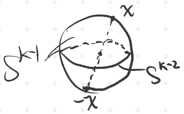
$C_i(\mathbb{RP}^n) \cong \mathbb{Z}$, $i = 0, 1, \cdots, n$; $C_i(\mathbb{RP}^n) = 0$ for $i > n$.

$H_0(\mathbb{RP}^n) \cong \mathbb{Z}$, $H_k(\mathbb{RP}^n) = 0$ for $k > n$.

$$0 \to \underset{\mathbb{Z}}{\underset{\|}{C_n}} \xrightarrow{d_n} \underset{\mathbb{Z}}{\underset{\|}{C_{n-1}}} \xrightarrow{d_{n-1}} \underset{\mathbb{Z}}{\underset{\|}{C_{n-2}}} \to \cdots \to \underset{\mathbb{Z}}{\underset{\|}{C_2}} \xrightarrow{d_2} \underset{\mathbb{Z}}{\underset{\|}{C_1}} \xrightarrow{d_1 = 0} \underset{\mathbb{Z}}{\underset{\|}{C_0}}$$

For $1 \le k \le n$, $d_k(e^k) = y_k \cdot e^{k-1}$, y_k is the degree of the composition

$$S^{k-1} \xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}.$$



Recall $\mathbb{RP}^{k-1} = S^{k-1}/x \sim -x$. $[x] = \{x, -x\}$.

$$\deg(q\varphi_k) = \deg(q\varphi_k)|_x + \deg(q\varphi_k)|_{-x}.$$

$S^{k-1} - S^{k-2} = \mathring{D}^{k-1}_+ \amalg \mathring{D}^{k-1}_-$

- The composition $q\varphi_k$ restricted to $\mathring{D}^{k-1}_+$ or $\mathring{D}^{k-1}_-$ is a homeomorphism

For a homeomorphism, its degree is ± 1.

Assume $\deg q\varphi_k|_{\mathring{D}^{k-1}_+} = 1$, $\quad (q\varphi_k)|_{\mathring{D}^{k-1}_-} = (q\varphi_k)|_{\mathring{D}^{k-1}_+} \circ (-1)$

$$S^{k-1} \xrightarrow{-1} S^{k-1} \xrightarrow{\varphi_k} \mathbb{RP}^{k-1} \xrightarrow{q} S^{k-1} \qquad \mathring{D}^{k-1}_- \xrightarrow{-1} \mathring{D}^{k-1}_+$$

$x \mapsto -x$

$q\varphi_k|_{\mathring{D}^{k-1}_-}$ is the restriction of the composition $(q\varphi_k \circ -1)$ to $\mathring{D}^{k-1}_+$

$$\deg(q\varphi_k)|_{\mathring{D}^{k-1}_-} = \deg q\varphi_k|_{\mathring{D}^{k-1}_+} \cdot \underset{(-1)^k}{\underset{\|}{\deg(-1)}} = (-1)^k.$$

Thus $d_k e^k = (1 + (-1)^k) \cdot e^{k-1}$.

- $$0 \to \underset{C_n}{\underset{\|}{\mathbb{Z}}} \xrightarrow{d_n} \mathbb{Z} \to \cdots \to \underset{C_2}{\underset{\|}{\mathbb{Z}}} \xrightarrow{2} \underset{C_1}{\underset{\|}{\mathbb{Z}}} \xrightarrow{d_1 = 0} \mathbb{Z} \to 0$$

$$\Rightarrow H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2 & \text{for } 1 \le k \le n,\ k \text{ odd} \\ \mathbb{Z} & \text{for } k = 0, \text{ or } k = n \text{ is even.} \\ 0 & \text{otherwise.} \end{cases}$$

Cor. $n = 2$, $\mathbb{RP}^2$ $\quad H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2 & k = 1 \\ \mathbb{Z} & k = 0 \\ 0 & k \ge 2 \end{cases}$

④ $M(\mathbb{Z}/m, n)$, $m, n \in \mathbb{N}$.

if $m=0$, $M(\mathbb{Z}, n) = S^n$

if $m>0$, $M(\mathbb{Z}/m, n) = S^n \cup_{\varphi_m} e^{n+1}$, $\varphi_m : S^n \to S^n$ has degree m.

$$\tilde{H}_k(M(\mathbb{Z}/m, n)) = \begin{cases} \mathbb{Z}/m, & k=n \\ 0 & k \neq n \end{cases}$$

proof : Exercise.

For a finitely generated abelian group $G \cong \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{r_i}$, p_i is a prime.

$$\text{Let } X = \left(\bigvee_{j=1}^{k} S_j^n\right) \vee \left(\bigvee_{i=1}^{l} M(\mathbb{Z}/p_i^{r_i}, n)\right) = M(G, n)$$

Moore space of type (G, n).

$$\text{Then } \tilde{H}_m(X) \cong \bigoplus_{j=1}^{k} \tilde{H}_m(S_j^n) \oplus \bigoplus_{i=1}^{l} \tilde{H}_m(M(\mathbb{Z}/p_i^{r_i}, n))$$

$$\cong \begin{cases} \mathbb{Z}^k \oplus \bigoplus_{i=1}^{l} \mathbb{Z}/p_i^{r_i} = G, & m=n \\ 0, & m \neq n. \end{cases}$$

Thus Every finitely generated abelian group can be realised as the reduced homology group of a CW complex.

Moore Space.

Homology groups with coefficients

$H_k(X; G)$, G abelian group

$$C_k(X) = \left\{ \textstyle\sum_i k_i \cdot \sigma_i \mid \sigma_i : \Delta^k \to X,\ k_i \in \mathbb{Z} \right\}$$

$\mathbb{Z}$ module

$$C_k(X; G) = \left\{ \textstyle\sum_i g_i \sigma_i \mid \sigma_i : \Delta^k \to X,\ g_i \in G \right\} \cong C_k(X) \otimes G$$

$$\partial_k\left(\textstyle\sum_i k_i \sigma_i\right) = \textstyle\sum_i k_i (\partial_k \sigma_i), \quad \partial_k^G\left(\textstyle\sum_i g_i \sigma_i\right) = \textstyle\sum_i g_i \cdot \partial_k \sigma_i$$

$$\partial_k \partial_{k+1} = 0 \Rightarrow \partial_k^G \partial_{k+1}^G = 0 \Longrightarrow H_k(X; G) = \operatorname{Ker} \partial_k^G / \operatorname{Im} \partial_{k+1}^G.$$

· All other properties hold for $H_k(X,A;G)$ with $H_k(X,A) = H_k(X,A;\mathbb{Z})$.

The 7 Eilenberg–Steenrod axioms still hold for $H_k(X,A;G)$.

· Cellular boundary formula

$$d_n e_\beta^n = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}, \quad d_{\alpha\beta} = \deg \Delta_{\alpha\beta}:$$
$$\Delta_{\alpha\beta}: S_\alpha^{n-1} \longrightarrow S_\beta^{n-1}.$$

Lemma. If $f: S^n \to S^n$ has degree m, then

$f_*: H_n(S^n; G) \longrightarrow H_n(S^n; G)$ is the multiplication by m.

$(f_* = \mu_m : g \mapsto mg)$.

proof. A homomorphism $\varphi: G \to H$ induces a homomorphism

$$\varphi_\# : C_k(X;G) \longrightarrow C_k(X;H)$$
$$\textstyle\sum_i g_i \sigma_i \longmapsto \sum_i \varphi(g_i)\, \sigma_i$$

$\partial \varphi_\# = \varphi_\# \partial$

and hence induces a homomorphism $\varphi_* : H_k(X;G) \to H_k(X;H)$.

$\varphi: \mathbb{Z} \to G$, $1 \mapsto g_0$

$$\begin{array}{ccc} H_n(S^n;\mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n;\mathbb{Z}) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ H_n(S^n;G) & \xrightarrow{f_*} & H_n(S^n;G) \cong \end{array} \qquad \begin{array}{ccc} 1 & \longmapsto & \deg f = m \\ \downarrow & & \downarrow \\ g & \xmapsto{f_* = \mu_m} & m \cdot g \end{array}$$

$\square$

Cor. $d_n : C_n(X;G) \longrightarrow C_{n-1}(X;G)$

$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta}^G\, e_\beta^{n-1}$, $d_{\alpha\beta}^G = \mu_{d_{\alpha\beta}}$, $d_{\alpha\beta}$ is the degree of $\Delta_{\alpha\beta}: S_\alpha^{n-1} \to S_\beta^{n-1}$. $\square$

Exercise. Compute

$$H_k(\mathbb{R}P^n;\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & k=0,1,\dots,n; \\ 0 & k>n. \end{cases}$$

Example.

$$H_k(S^n;G) = \begin{cases} G & k=0,n \\ 0 & k\neq 0,n \end{cases}.$$

$\square$